# STAT 115:Experimental Designs

Multisample inference:
Analysis of Variance

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# Learning Objectives

1. Describe Analysis of Variance (ANOVA)
2. Explain the Rationale of ANOVA
3. Compare Experimental Designs
4. Test the Equality of 2 or More Means
   - Completely Randomized Design
   - Randomized Block Design

# Analysis of Variance

An *analysis of variance* is a technique that partitions the total sum of squares of deviations of the observations about their mean into portions associated with independent variables in the experiment and a portion associated with error

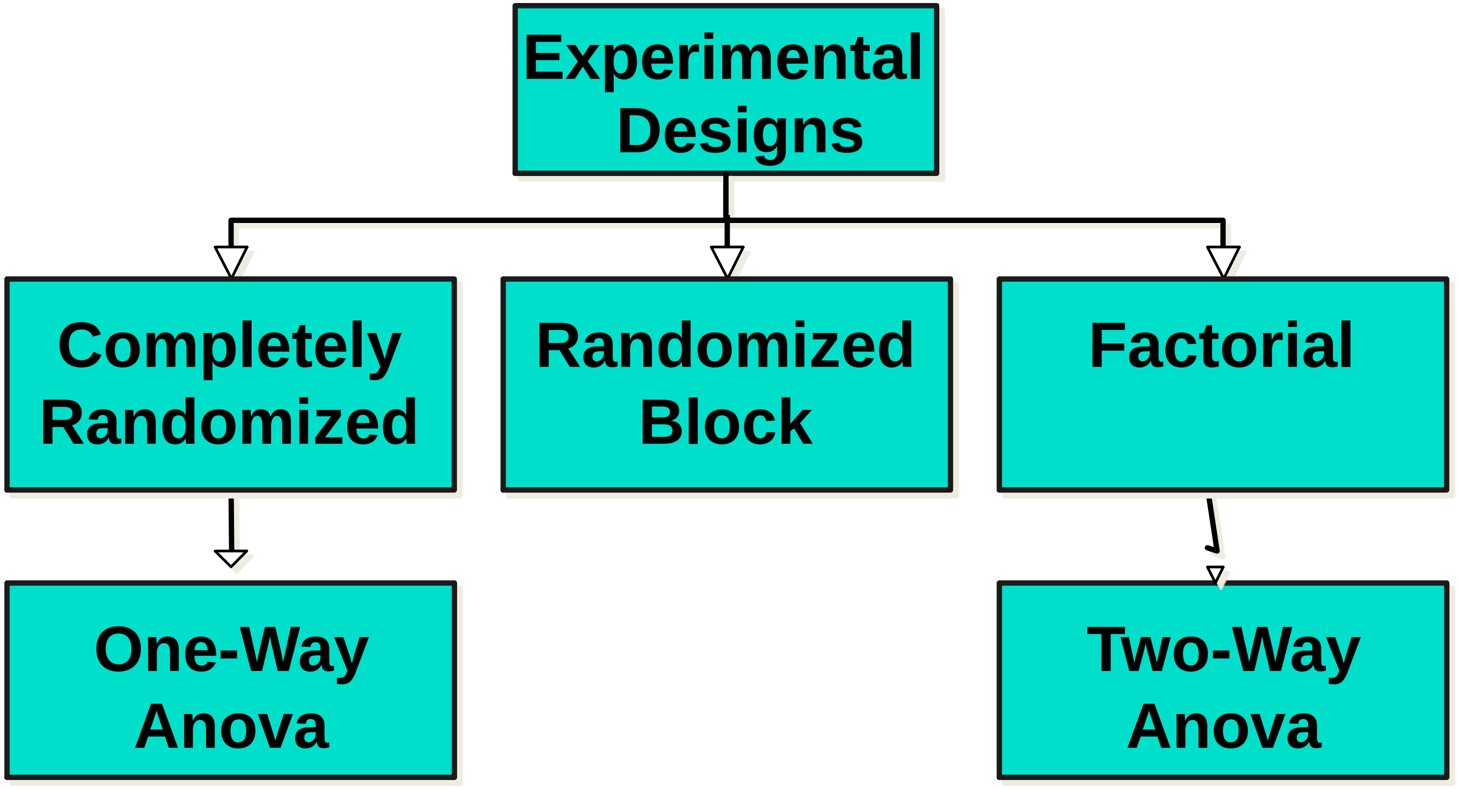
# Analysis of Variance

A *factor* refers to a categorical quantity under examination in an experiment as a possible cause of variation in the response variable.

# Analysis of Variance

*Levels* refer to the categories, measurements, or strata of a factor of interest in the experiment.

# Types of Experimental Designs



# Completely Randomized Design

# Completely Randomized Design

1. Experimental Units (Subjects) Are Assigned Randomly to Treatments
   – Subjects are Assumed Homogeneous

2. One Factor or Independent Variable
   – 2 or More Treatment Levels or groups

3. Analyzed by One-Way ANOVA

# One-Way ANOVA F-Test

1. Tests the Equality of 2 or More (*p*) Population Means

2. Variables
   – One Nominal Independent Variable
   – One Continuous Dependent Variable

# One-Way ANOVA F-Test Assumptions

1. Randomness & Independence of Errors
2. Normality
   - Populations (for each condition) are Normally Distributed
3. Homogeneity of Variance
   - Populations (for each condition) have Equal Variances

# One-Way ANOVA F-Test Hypotheses

**$H_0$: $\mu_1 = \mu_2 = \mu_3 = \ldots = \mu_p$**

- **All Population Means are Equal**
- **No Treatment Effect**

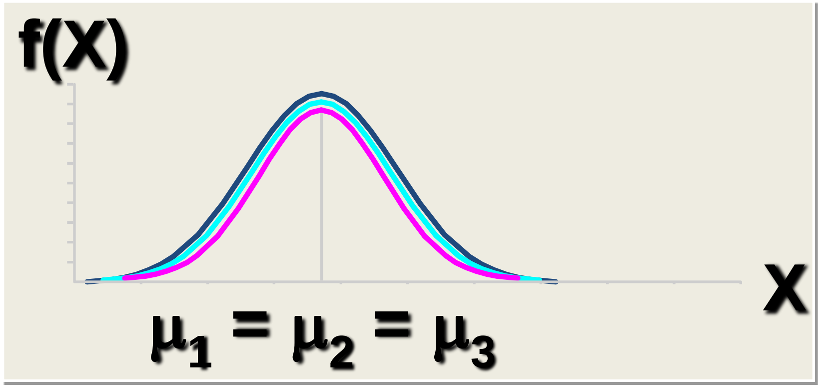
**$H_a$: At Least 1 Pop. Mean is Different**

- **Treatment Effect**
- NOT $\mu_1 \neq \mu_2 \neq \ldots \neq \mu_p$

# One-Way ANOVA F-Test Hypotheses

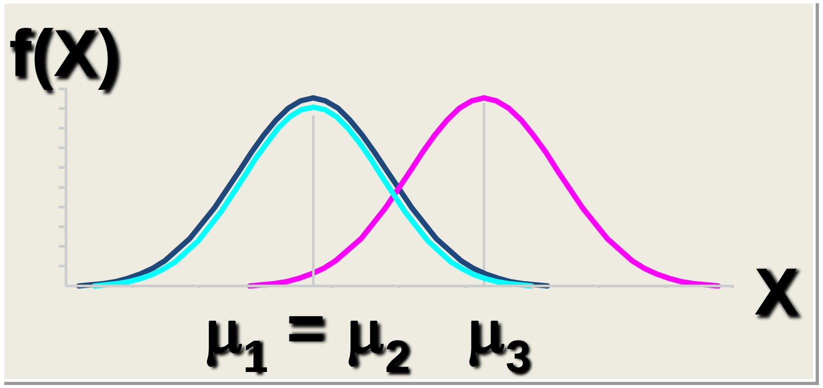
**$H_0$: $\mu_1 = \mu_2 = \mu_3 = \ldots = \mu_p$**

- **All Population Means are Equal**
- **No Treatment Effect**




**$H_a$: At Least 1 Pop. Mean is Different**

- **Treatment Effect**



# One-Way ANOVA
# Basic Idea

1. Compares 2 Types of Variation to Test

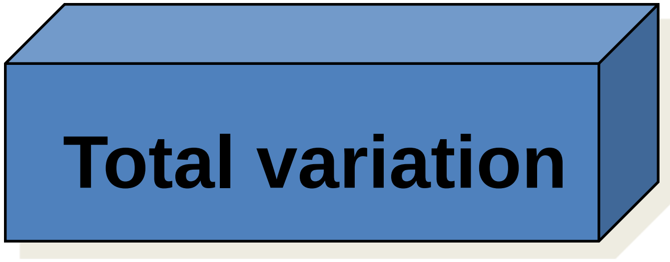
   Equality of Means

2. If Treatment Variation Is Significantly Greater Than Random Variation then Means Are **Not** Equal

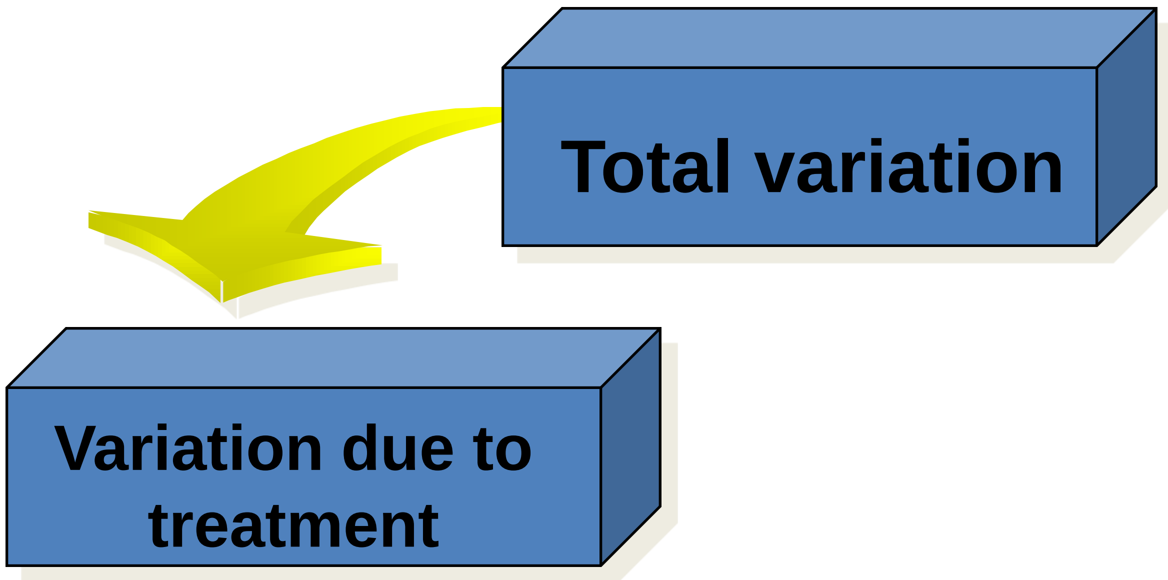
3. Variation Measures Are Obtained by 'Partitioning' Total Variation

# One-Way ANOVA
# Partitions Total Variation

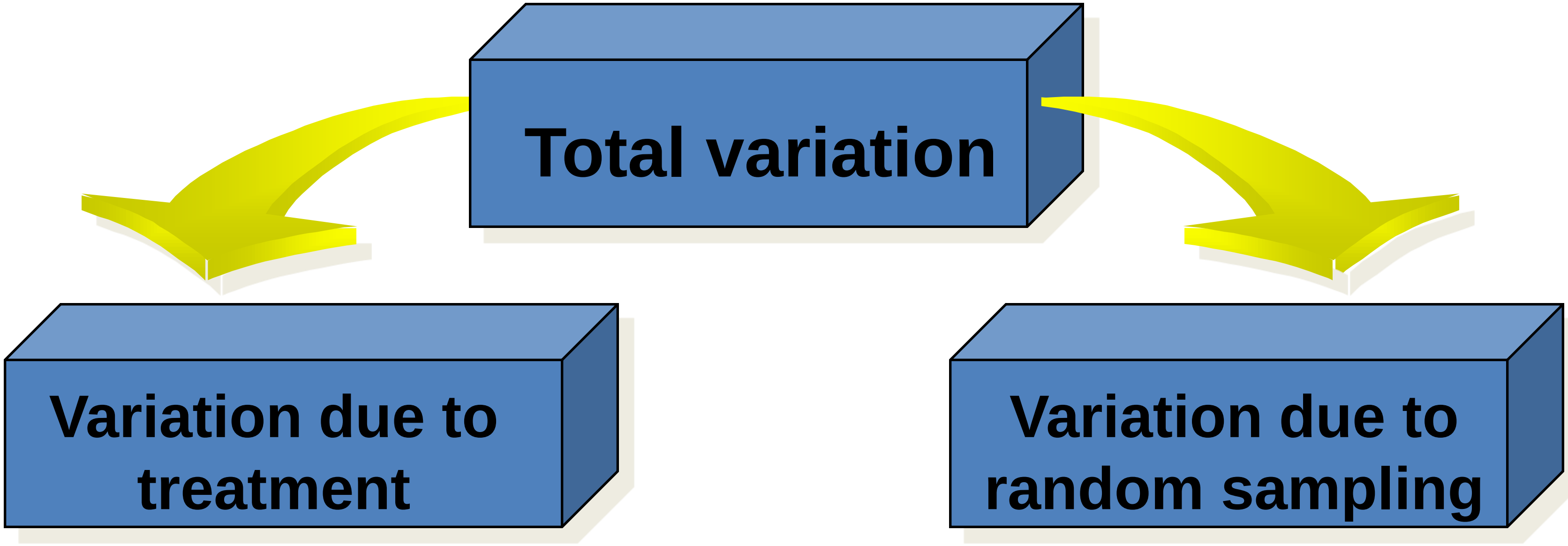
# One-Way ANOVA
# Partitions Total Variation



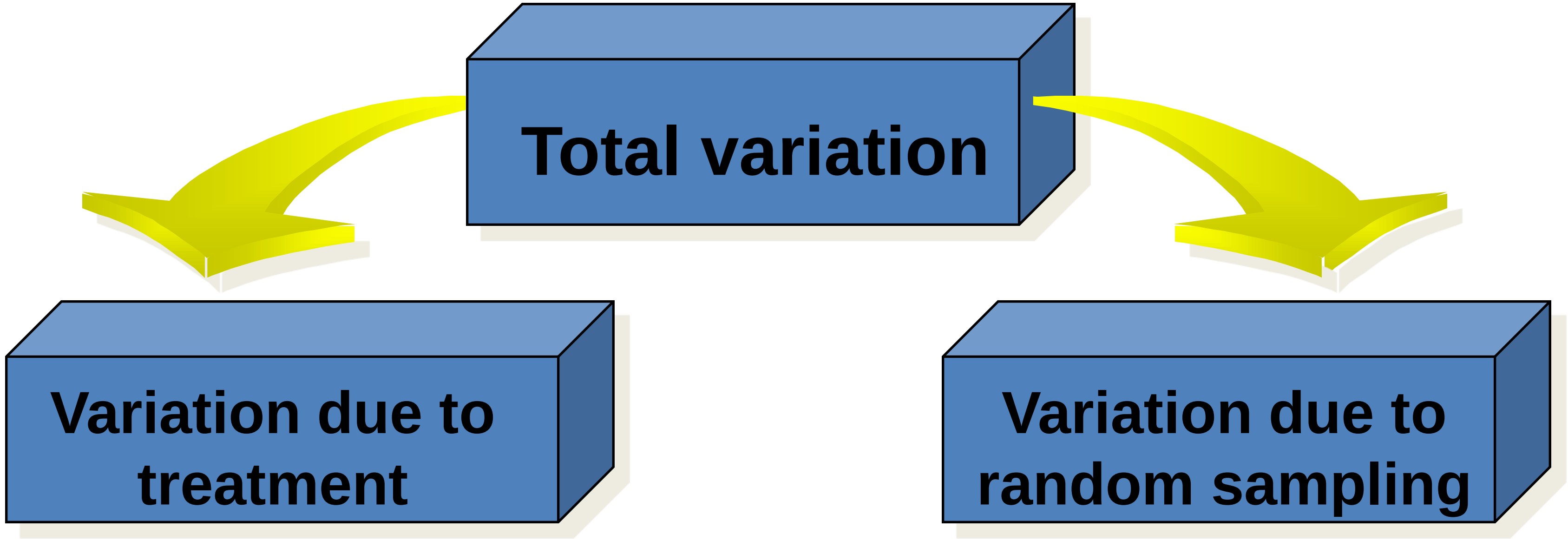
# One-Way ANOVA
# Partitions Total Variation



# One-Way ANOVA
# Partitions Total Variation



# One-Way ANOVA
# Partitions Total Variation



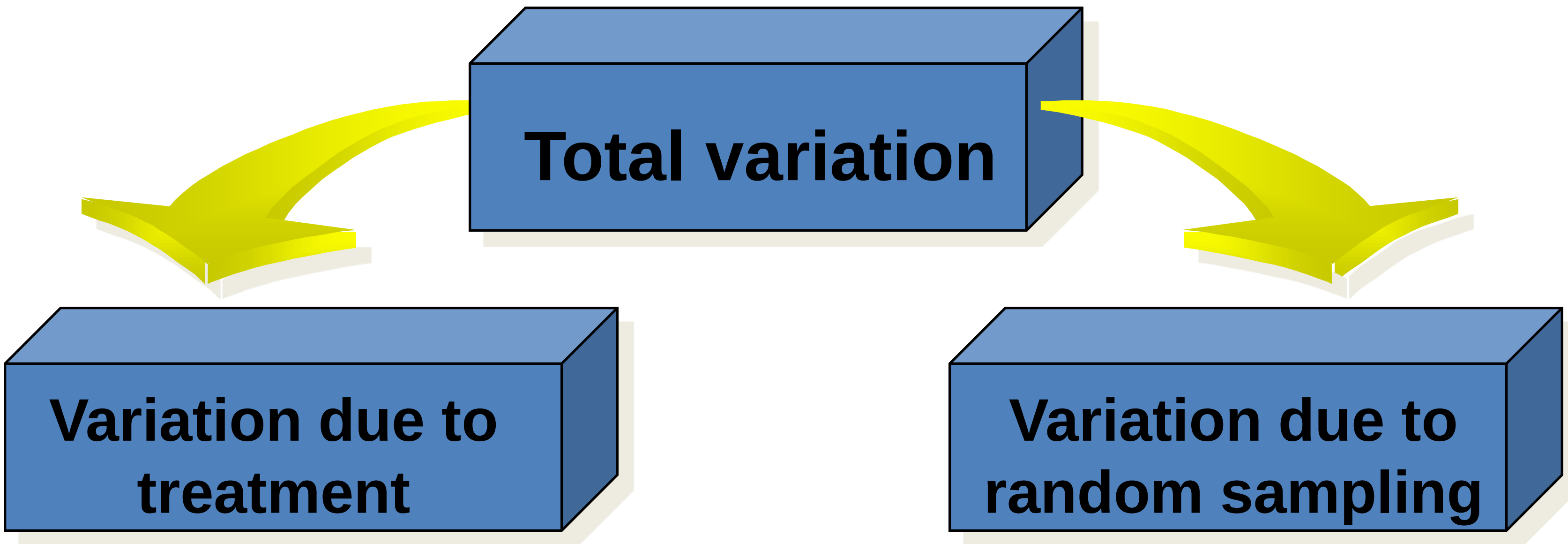

Sum of Squares Among

Sum of Squares Between

Sum of Squares Treatment

Among Groups Variation

# One-Way ANOVA
# Partitions Total Variation

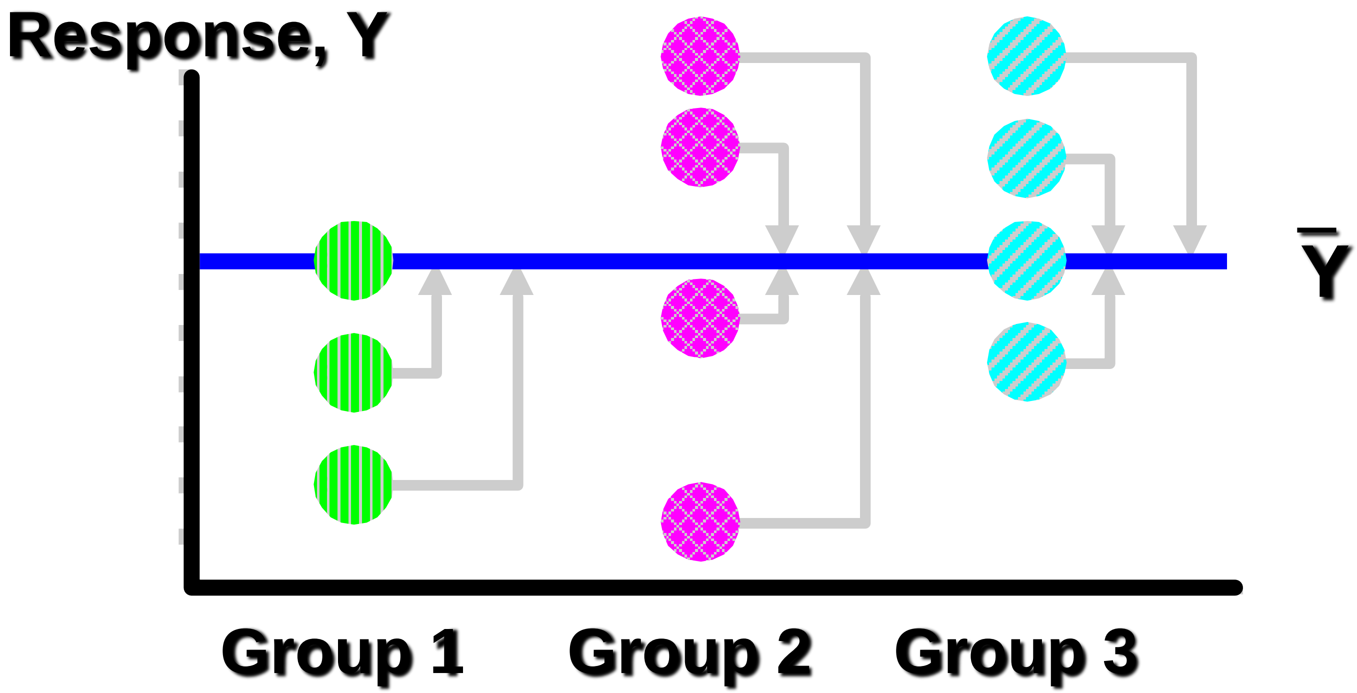



Sum of Squares Among
Sum of Squares Between
**Sum of Squares Treatment (SST)**
Among Groups Variation

Sum of Squares Within
**Sum of Squares Error (SSE)**
Within Groups Variation

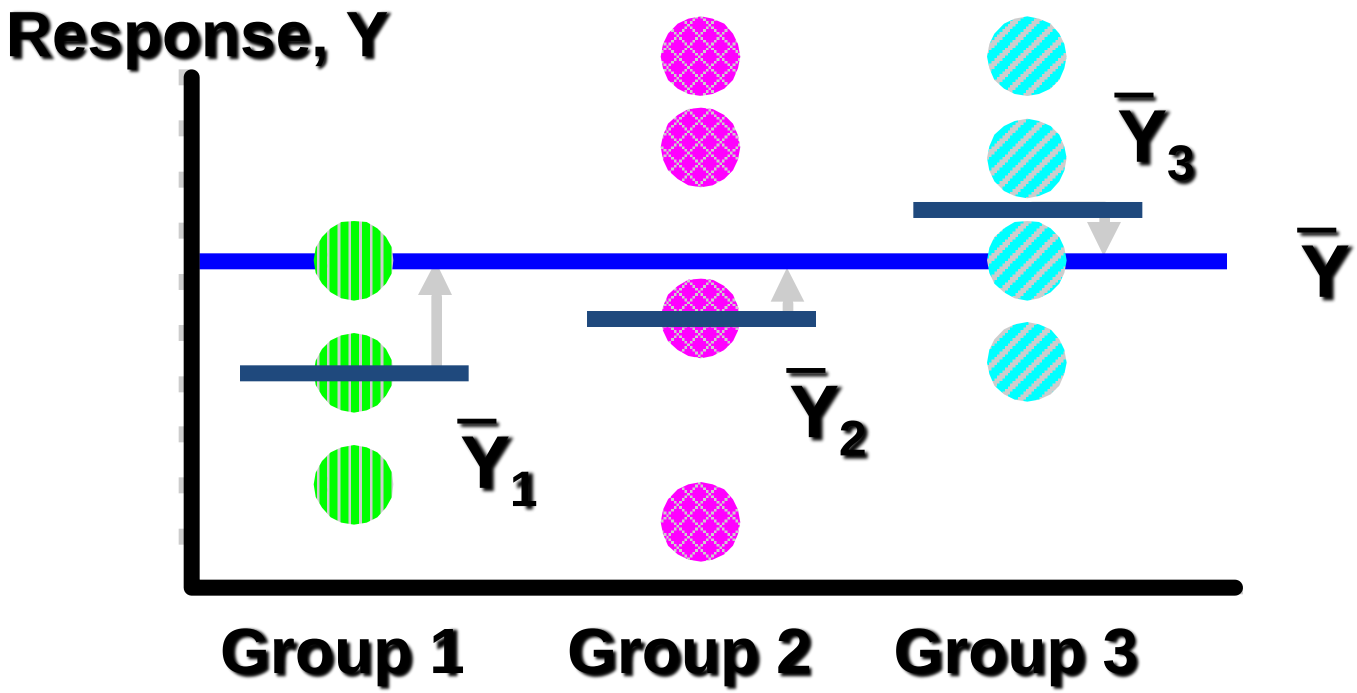
# Total Variation

$$SS(Total) = (Y_{11} - \bar{Y})^2 + (Y_{21} - \bar{Y})^2 + \ldots + (Y_{ij} - \bar{Y})^2$$



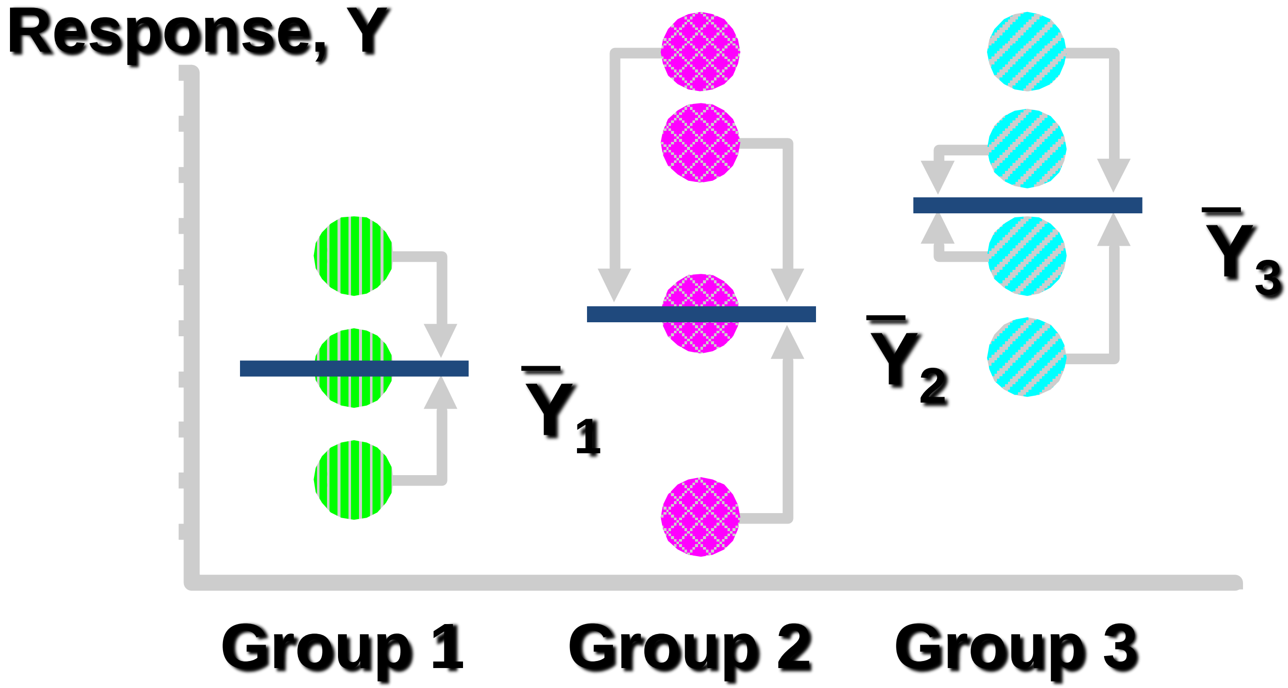
# Treatment Variation

$$SST = n_1(\bar{Y}_1 - \bar{Y})^2 + n_2(\bar{Y}_2 - \bar{Y})^2 + \ldots + n_p(\bar{Y}_p - \bar{Y})^2$$



# Random (Error) Variation

$$SSE = \left(Y_{11} - \bar{Y}_1\right)^2 + \left(Y_{21} - \bar{Y}_1\right)^2 + \ldots + \left(Y_{pj} - \bar{Y}_p\right)^2$$



# One-Way ANOVA F-Test Test Statistic

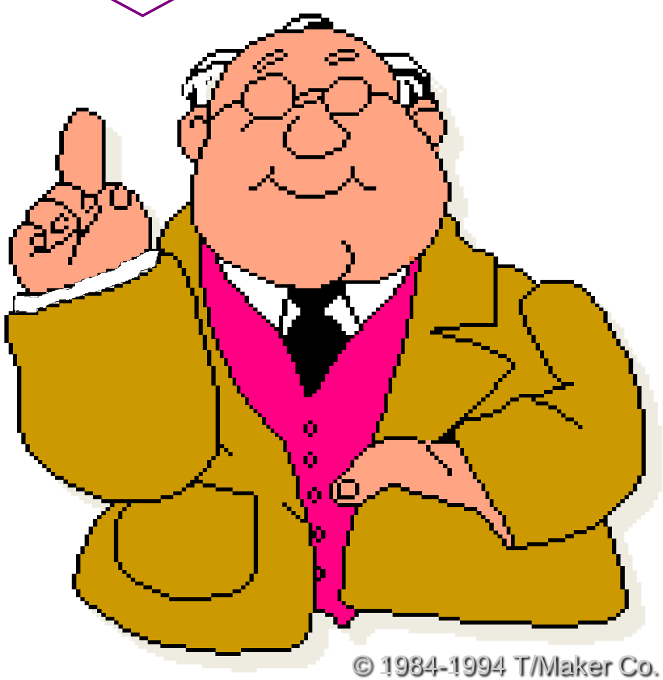
- 1. Test Statistic
  - *F* = *MST* / *MSE* $= \dfrac{STT/(p-1)}{SSE/(n-p)}$
    - *MST* Is Mean Square for Treatment
    - *MSE* Is Mean Square for Error
- 2. Degrees of Freedom
  - $\nu_1 = p - 1$
  - $\nu_2 = n - p$
    - *p* = # Populations, Groups, or Levels
    - *n* = Total Sample Size

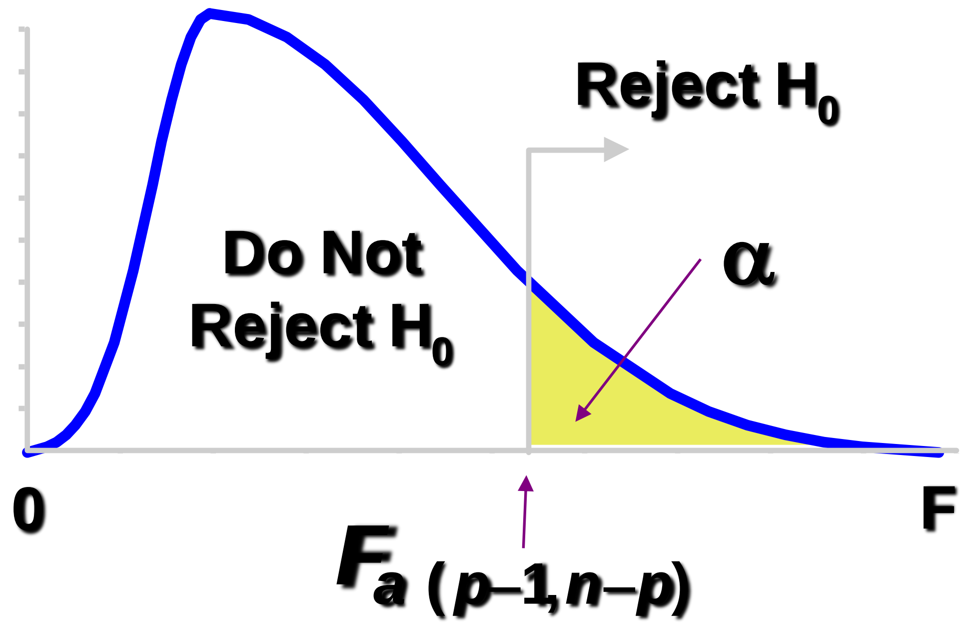
# One-Way ANOVA Summary Table

| Source of Variation | Degrees of Freedom | Sum of Squares | Mean Square (Variance) | F |
|---|---|---|---|---|
| Treatment | p - 1 | SST | MST = SST/(p - 1) | $\frac{MST}{MSE}$ |
| Error | n - p | SSE | MSE = SSE/(n - p) | |
| Total | n - 1 | SS(Total) = SST+SSE | | |

# One-Way ANOVA F-Test Critical Value

**If means are equal,**
***F* = *MST* / *MSE* ≈ 1.**
**Only reject large *F*!**




**Always One-Tail!**

# One-Way ANOVA F-Test Example

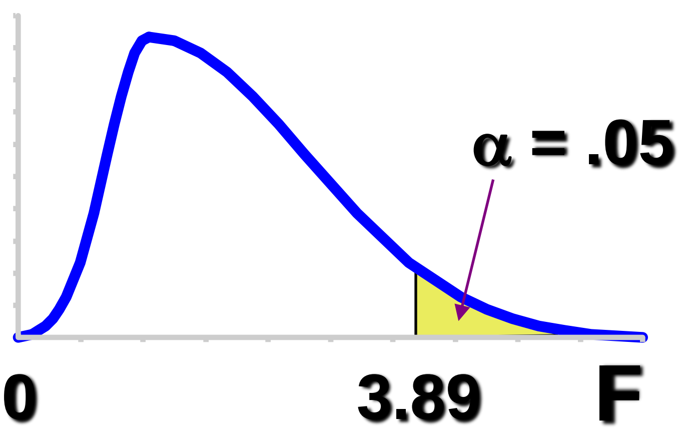
As a vet epidemiologist you want to see if 3 food supplements have different mean milk yields. You assign 15 cows, 5 per food supplement.

Question: At the .05 level, is there a difference in mean yields?

| Food1 | Food2 | Food3 |
|---|---|---|
| 25.40 | 23.40 | 20.00 |
| 26.31 | 21.80 | 22.20 |
| 24.10 | 23.50 | 19.75 |
| 23.74 | 22.75 | 20.60 |
| 25.10 | 21.60 | 20.40 |

# One-Way ANOVA F-Test Solution

- $H_0$: $\mu_1 = \mu_2 = \mu_3$
- $H_a$: Not All Equal
- $\alpha = .05$
- $\nu_1 = 2$ $\nu_2 = 12$
- Critical Value(s):




**Test Statistic:**

$$F = \frac{MST}{MSE} = \frac{23.5820}{.9211} = 25.6$$

**Decision:**

Reject at $\alpha = .05$

**Conclusion:**

There Is Evidence Pop. Means Are Different

# Summary Table Solution

| Source of Variation | Degrees of Freedom | Sum of Squares | Mean Square (Variance) | F |
|---|---|---|---|---|
| Food | 3 - 1 = 2 | 47.1640 | 23.5820 | 25.60 |
| Error | 15 - 3 = 12 | 11.0532 | .9211 | |
| Total | 15 - 1 = 14 | 58.2172 | | |

# SAS CODES FOR ANOVA

- **Data** Anova;
- input group$ milk @@;
- cards;
- food1 25.40 food2 23.40 food3 20.00
- food1 26.31 food2 21.80 food3 22.20
- food1 24.10 food2 23.50 food3 19.75
- food1 23.74 food2 22.75 food3 20.60
- food1 25.10 food2 21.60 food3 20.40
- ;
- **run**;

- **proc anova**; /* or PROC GLM */
- class group;
- model milk=group;
- **run**;

# SAS OUTPUT - ANOVA

| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Model | 2 | 47.16400000 | 23.58200000 | 25.60 | <.0001 |
| Error | 12 | 11.05320000 | 0.92110000 | | |
| Corrected Total | 14 | 58.21720000 | | | |

# Pair-wise comparisons

- Needed when the overall F test is rejected
- Can be done without adjustment of type I error if other comparisons were planned in advance (least significant difference - LSD method)
- Type I error needs to be adjusted if other comparisons were not planned in advance (Bonferroni's and scheffe's methods)

# Fisher's Least Significant Difference (LSD) Test

To compare level 1 and level 2

$$t = \left(\bar{y}_1 - \bar{y}_2\right) \Big/ \sqrt{MSE\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

*Compare this to* $t_{\alpha/2}$ = Upper-tailed value or - $t_{\alpha/2}$ *lower-*tailed from Student's t-distribution for $\alpha/2$ and (n - p) degrees of freedom

MSE = Mean square within from ANOVA table

*n* = Number of subjects

p = Number of levels

# Bonferroni's method

To compare level 1 and level 2

$$t = \left(\bar{y}_1 - \bar{y}_2\right) \Big/ \sqrt{MSE\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

Adjust the significance level α by taking the new significance level α*

$$\alpha^* = \alpha / \binom{p}{2}$$

# SAS CODES FOR multiple comparisons

```
proc anova;
class group;
model milk=group;
means group/ lsd bon;
run;
```

# SAS OUTPUT - LSD

t Tests (LSD) for milk

NOTE: This test controls the Type I comparisonwise error rate, not the experimentwise error rate.

| | |
|---|---|
| Alpha | 0.05 |
| Error Degrees of Freedom | 12 |
| Error Mean Square | 0.9211 |
| Critical Value of t | 2.17881 = $t_{.975,12}$ |
| Least Significant Difference | 1.3225 |

Means with the same letter are not significantly different.

| t Grouping | Mean | N | group |
|---|---|---|---|
| A | 24.9300 | 5 | food1 |
| B | 22.6100 | 5 | food2 |
| C | 20.5900 | 5 | food3 |

# SAS OUTPUT - Bonferroni

Bonferroni (Dunn) t Tests for milk

NOTE: This test controls the Type I experimentwise error rate

| | |
|---|---|
| Alpha | 0.05 |
| Error Degrees of Freedom | 12 |
| Error Mean Square | 0.9211 |
| Critical Value of t | 2.77947=$t_{1-0.05/3/2,12}$ |
| Minimum Significant Difference | 1.6871 |

Means with the same letter are not significantly different.

| Bon Grouping | Mean | N | group |
|---|---|---|---|
| A | 24.9300 | 5 | food1 |
| B | 22.6100 | 5 | food2 |
| C | 20.5900 | 5 | food3 |

# Randomized Block Design

# Randomized Block Design

1. Experimental Units (Subjects) Are Assigned Randomly within Blocks
   – Blocks are Assumed Homogeneous
2. One Factor or Independent Variable of Interest
   – 2 or More Treatment Levels or Classifications
3. One Blocking Factor

# Randomized Block Design

| Factor Levels: (Treatments) | A, B, C, D | | | |
|---|---|---|---|---|
| Experimental Units | Treatments are randomly assigned within blocks | | | |
| Block 1 | A | C | D | B |
| Block 2 | C | D | B | A |
| Block 3 | B | A | D | C |
| . . . | . . . | . . . | . . . | . . . |
| Block b | D | C | A | B |

# Randomized Block F-Test

1. Tests the Equality of 2 or More ($p$) Population Means

2. Variables
   - One Nominal Independent Variable
   - One Nominal Blocking Variable
   - One Continuous Dependent Variable

# Randomized Block F-Test Assumptions

1. Normality
   – Probability Distribution of each Block-Treatment combination is Normal

2. Homogeneity of Variance
   – Probability Distributions of all Block-Treatment combinations have Equal Variances

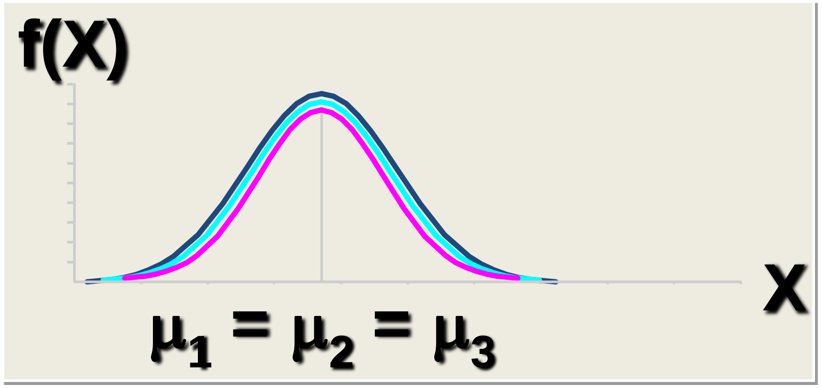
# Randomized Block F-Test Hypotheses

- **$H_0$: $\mu_1 = \mu_2 = \mu_3 = ... = \mu_p$**
  - **All Population Means are Equal**
  - **No Treatment Effect**
- **$H_a$: At Least 1 Pop. Mean is Different**
  - **Treatment Effect**
  - **$\mu_1 \neq \mu_2 \neq ... \neq \mu_p$ Is wrong**

# Randomized Block F-Test Hypotheses

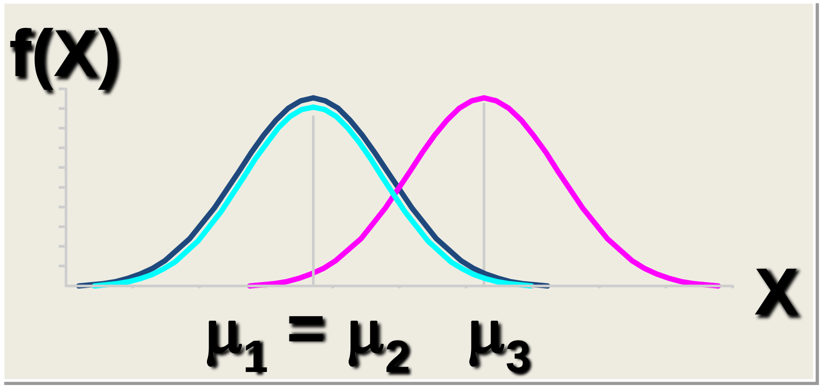
$H_0$: $\mu_1 = \mu_2 = \ldots = \mu_p$

- All Population Means are Equal
- No Treatment Effect




$H_a$: At Least 1 Pop. Mean is Different

- Treatment Effect
- $\mu_1 \neq \mu_2 \neq \ldots \neq \mu_p$ Is **wrong**



# The F Ratio for Randomized Block Designs

- SS=SSE+SSB+SST

$$F = \frac{\mathrm{MST}}{\mathrm{MSE}} = \frac{SST/(p-1)}{SSE/(n-1-p+1-b+1)}$$

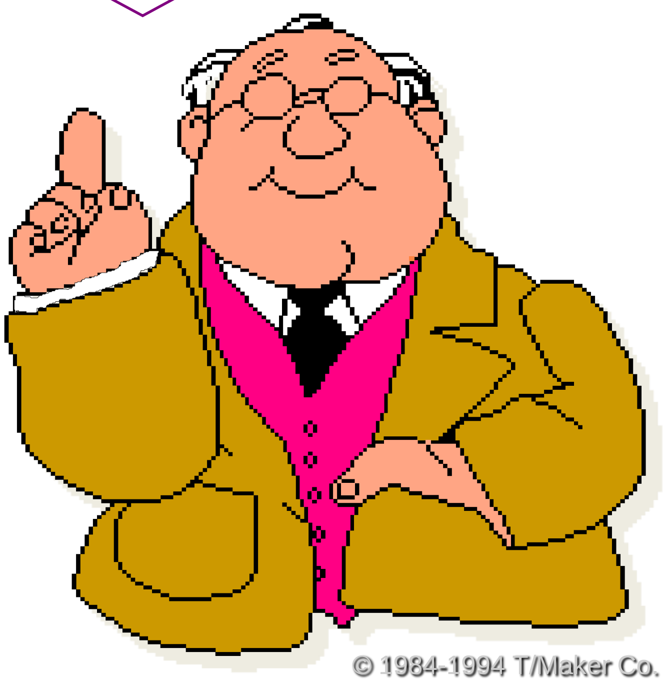
$$= \frac{SST/(p-1)}{SSE/(n-p-b+1)}$$

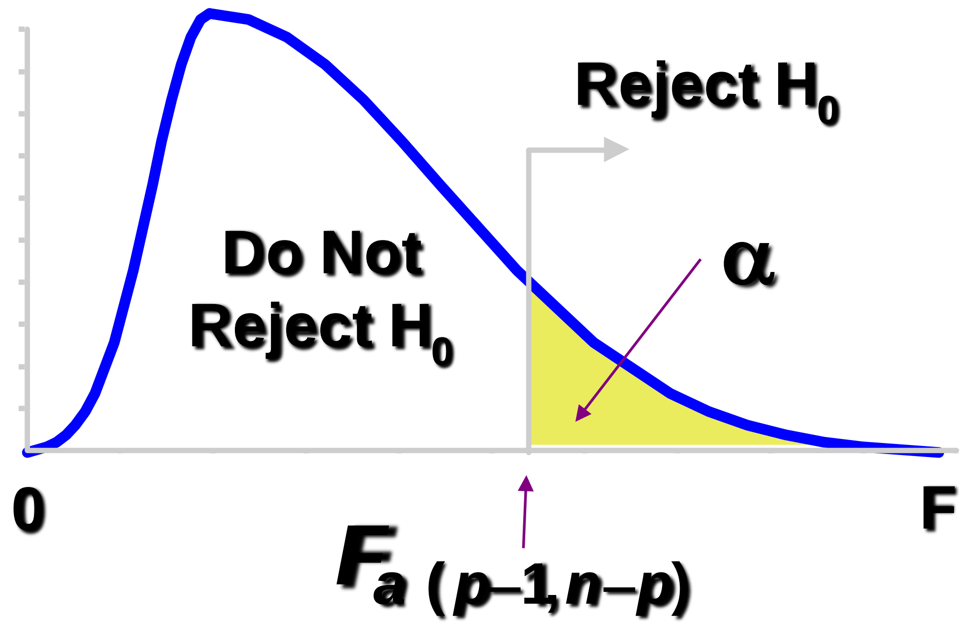
# Randomized Block F-Test
# Test Statistic

- 1. Test Statistic
  - $F = MST / MSE$
    - *MST* Is Mean Square for Treatment
    - *MSE* Is Mean Square for Error
- 2. Degrees of Freedom
  - $\nu_1 = p - 1$
  - $\nu_2 = n - b - p + 1$
    - *p* = # Treatments, *b* = # Blocks, *n* = Total Sample Size

# Randomized Block F-Test Critical Value

**If means are equal,**
***F*** **=** ***MST / MSE*** **≈ 1.**
**Only reject large** ***F*!**




**Always One-Tail!**

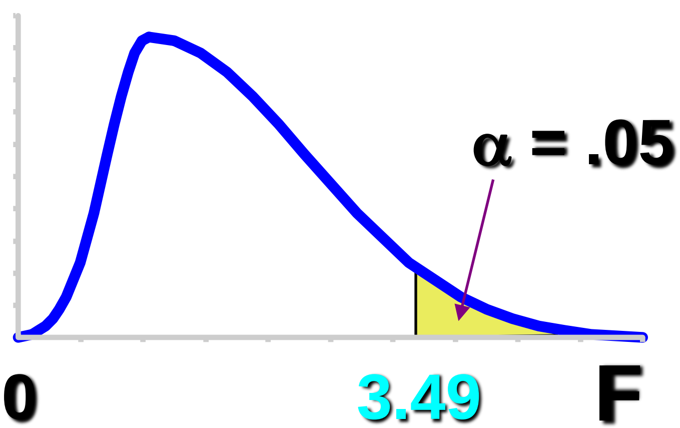
# Randomized Block F-Test Example

- You wish to determine which of four brands of tires has the longest tread life. You randomly assign one of each brand (A, B, C, and D) to a tire location on each of 5 cars. At the .05 level, is there a difference in mean tread life?

| | Tire Location | | | |
|---|---|---|---|---|
| Block | Left Front | Right Front | Left Rear | Right Rear |
| Car 1 | A: 42,000 | C: 58,000 | B: 38,000 | D: 44,000 |
| Car 2 | B: 40,000 | D: 48,000 | A: 39,000 | C: 50,000 |
| Car 3 | C: 48,000 | D: 39,000 | B: 36,000 | A: 39,000 |
| Car 4 | A: 41,000 | B: 38,000 | D: 42,000 | C: 43,000 |
| Car 5 | D: 51,000 | A: 44,000 | C: 52,000 | B: 35,000 |

# Randomized Block F-Test Solution

- H0: $\mu_1 = \mu_2 = \mu_3 = \mu_4$
- Ha: Not All Equal
- $\alpha$ = .05
- $\nu_1 = 3$  $\nu_2 = 12$
- Critical Value(s):




**Test Statistic:**

***F* = 11.9933**

**Decision:**

**Reject at $\alpha$ = .05**

**Conclusion:**

**There Is Evidence Pop. Means Are Different**

# SAS CODES FOR ANOVA

```
data block;
input Block$ trt$ resp @@;
cards;
Car1        A: 42000 Car1 C: 58000 Car1  B: 38000 Car1 D: 44000
Car2        B: 40000 Car2 D: 48000 Car2  A: 39000 Car2 C: 50000
Car3        C: 48000 Car3 D: 39000 Car3  B: 36000 Car3 A: 39000
Car4        A: 41000 Car4 B: 38000 Car4  D: 42000 Car4 C: 43000
Car5        D: 51000 Car5 A: 44000 Car5  C: 52000 Car5 B: 35000
;
run;

proc anova;
class trt block;
model resp=trt block;
Means trt /lsd bon;
run;
```

# SAS OUTPUT - ANOVA

Dependent Variable: resp

| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Model | 7 | 544550000.0 | 77792857.1 | 6.22 | 0.0030 |
| Error | 12 | 150000000.0 | 12500000.0 | | |
| Corrected Total | 19 | 694550000.0 | | | |

| R-Square | Coeff Var | Root MSE | resp Mean |
|---|---|---|---|
| 0.784033 | 8.155788 | 3535.534 | 43350.00 |

| Source | DF | Anova SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| trt | 3 | 449750000.0 | 149916666.7 | 11.99 | 0.0006 |
| Block | 4 | 94800000.0 | 23700000.0 | 1.90 | 0.1759 |

# SAS OUTPUT - LSD

Means with the same letter are not significantly different.

| t Grouping | | Mean | N | trt |
|---|---|---|---|---|
| | A | 50200 | 5 | C: |
| | B | 44800 | 5 | D: |
| | B | | | |
| C | B | 41000 | 5 | A: |
| C | | | | |
| C | | 37400 | 5 | B: |

# SAS OUTPUT - Bonferroni

Means with the same letter are not significantly different.

| Bon Grouping | | Mean | N | trt |
|---|---|---|---|---|
| | A | 50200 | 5 | C: |
| | A | | | |
| B | A | 44800 | 5 | D: |
| B | | | | |
| B | C | 41000 | 5 | A: |
| | C | | | |
| | C | 37400 | 5 | B: |