Section 6.1

Summation

The Summation Notation (page 186)

$$\sum_{i=1}^{n} X_i = X_1 + X_2 + ... + X_n$$

where Σ (capital Greek letter sigma) is the summation notation

X_i is the value of the variable for the ith observation

(the expression to the right of Σ is the term of the summation)

i (letter below Σ, left of = sign) is the index of the summation

1 (number below Σ, right of = sign) is the lower limit of summation

n (number above Σ) is the upper limit of summation

Index set = collection of consecutive integers from lower limit to upper limit of summation

Expansion of the Summation Notation

- The terms of the sum are determined by successively replacing the index in the term of the summation by the elements in the index set.

Examples:

$$\sum_{i=1}^{3} X_i = X_1 + X_2 + X_3$$

$$\sum_{i=2}^{3} X^i = X^2 + X^3$$

$$\sum_{j=2}^{3} X^j = X^2 + X^3$$

Example 6.1. (pages 186-187)

The weights in pounds of the five students are as follows: 110, 90, 105, 120, and 115. Express the formula for the total weight of these students using the summation notation.

Solution: Let X = weight of a student in pounds
n = 5 students

X_1 represents the weight of the first student = 110 lbs.
X_2 represents the weight of the second student = 90 lbs.
X_3 represents the weight of the third student = 105 lbs.
X_4 represents the weight of the fourth student= 120 lbs.
X_5 represents the weight of the fifth student = 115 lbs.

We express the formula for the total weight of the 5 students in a compact manner using the summation notation as $\sum_{i=1}^{5} X_i$. This notation means

$$\sum_{i=1}^{5} X_i = X_1 + X_2 + X_3 + X_4 + X_5 = 110 + 90 + 105 + 120 + 115 = 540 \text{ lbs.}$$

Some Notes on Summation (page 187)

(1) The index may be any letter, but the letters i, j, and k are the most commonly used. Example:

$$\sum_{i=1}^{n} X_i = \sum_{j=1}^{n} X_j$$

(2) The lower limit of the summation may start with any integer smaller than the upper limit. Example:

$$\sum_{k=3}^{6} X_k = X_3 + X_4 + X_5 + X_6$$

(3) The index will not necessarily appear as a subscript in the term of the summation. Example:

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5$$

Example 6.2 (page 187)

b) Find $\sum_{j=1}^{3} Y_j^2$ when $Y_1 = 2,\ Y_2 = 5,\ Y_3 = -2$

$$\sum_{j=1}^{3} Y_j^2 = Y_1^2 + Y_2^2 + Y_3^2 = 2^2 + 5^2 + (-2)^2 = 33$$

c) Evaluate $\sum_{x=1}^{4} X$.

$$\sum_{x=1}^{4} X = 1 + 2 + 3 + 4 = 10$$

e) Find $\sum_{i=1}^{3} (X_i - i)$ when $X_1 = 3,\ X_2 = 5,\ X_3 = 7$

$$\sum_{i=1}^{3} (X_i - i) = (X_1 - 1) + (X_2 - 2) + (X_3 - 3)$$

$$= (3 - 1) + (5 - 2) + (7 - 3) = 9$$

g) Find $\sum_{i=1}^{3} X_i Y_i$

when $X_1 = 2,\ X_2 = 1,\ X_3 = 4,\ Y_1 = 1,\ Y_2 = 3,\ Y_3 = -1$

$$\sum_{i=1}^{3} X_i Y_i = X_1 Y_1 + X_2 Y_2 + X_3 Y_3$$

$$= (2)(1) + (1)(3) + (4)(-1) = 1$$

h) Find $\left(\sum_{i=1}^{3} X_i\right)\left(\sum_{i=1}^{3} Y_i\right)$

when $X_1 = 2, X_2 = 1, X_3 = 4, Y_1 = 1, Y_2 = 3, Y_3 = -1$

$$\left(\sum_{i=1}^{3} X_i\right)\left(\sum_{i=1}^{3} Y_i\right) = (X_1 + X_2 + X_3)(Y_1 + Y_2 + Y_3)$$

$$= (2+1+4)(1+3+-1) = 21$$

Exercise #1 (page 191)

i	1	2	3	4	5
X_i	2	3	4	5	6
Y_i	1	4	2	3	5

Given the data above, find the following:

e) $\sum_{i=1}^{5} X_i^2$

f) $\left(\sum_{i=1}^{5} X_i\right)^2$

g) $\sum_{i=1}^{5} (X_i - Y_i)^2$

h) $\sum_{i=3}^{5} (X_i - 1)$

Exercise

Write the expansion of the following summation:

a) $\sum_{i=3}^{6} \frac{Z_i}{Y_i}$

b) $\frac{\sum_{i=3}^{6} Z_i}{\sum_{i=3}^{6} Y_i}$

c) $\sum_{k=0}^{3} (Y_j^k - Y_k)$

d) $\sum_{l=2}^{4} \sqrt{i}$

e) $\sum_{j=1}^{3} 3X_j^j$

Additional Notes on Summation (page 190-191)

(1) $\sum_{i=1}^{n} X_i^2 \neq \left(\sum_{i=1}^{n} X_i\right)^2$

(2) $\sum_{i=1}^{n} (X_i + Y_i)^2 \neq \sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} Y_i^2$

(3) $\sum_{i=1}^{n} X_i Y_i \neq \left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} Y_i\right)$

(4) $\sum_{i=1}^{n} \frac{X_i}{Y_i} \neq \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} Y_i}$

(5) $\sum_{i=1}^{n} \sqrt{X_i} \neq \sqrt{\sum_{i=1}^{n} X_i}$

Some Properties of Summation (pages 188-189)

(1) The summation of the sum (or difference) of two or more terms equals the sum (or difference) of the individual summations. For example,

$$\sum_{i=1}^{n} (X_i + Y_i + Z_i) = \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} Z_i$$

$$\sum_{i=1}^{n} (X_i - Y_i - Z_i) = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} Z_i$$

(2) The summation of the product of a constant, c, with X_i, equals the product of the constant with the summation of X_i, i.e.,

$$\sum_{i=1}^{n} cX_i = c\sum_{i=1}^{n} X_i$$

(3) The summation of a constant, c, with index set = {1,2,…,n}, equals the product of n and c, i.e.,

$$\sum_{i=1}^{n} c = nc$$

Example

$$\sum_{j=1}^{6}(X_j+2)^2 = \sum_{j=1}^{6}(X_j^2+4X_j+4)$$

$$= \sum_{j=1}^{6}X_j^2 + \sum_{j=1}^{6}4X_j + \sum_{j=1}^{6}4$$

$$= \sum_{j=1}^{6}X_j^2 + 4\sum_{j=1}^{6}X_j + (6)(4)$$

$$= \sum_{j=1}^{6}X_j^2 + 4\sum_{j=1}^{6}X_j + 24$$

Assignment: (page 192)

Exercise #2 a – d.

Exercise #3d. Define $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ **and** $\overline{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$. **This is called the sample mean.**