Mathematics 53: Exercises on Derivatives, Tangent Lines, Normal Lines, Chain Rule, Implicit Differentiation and Differentiability

I. Use the definition to find the derivative of the following functions.

1. $f(x) = 3x^2 - x - 2$
2. $g(x) = \sqrt{2x-2}$
3. $h(x) = \dfrac{1}{x+2}$
4. $m(x) = \tan x$

II. Find $\dfrac{dy}{dx}$. There is no need to simplify.

1. $y = \dfrac{x+2}{\sec x}$
2. $y = \dfrac{\frac{x^2}{4} + \sqrt[3]{x^2}}{x\cos 2x}$
3. $y = \left[\dfrac{2x^3+3x^2-x}{\sqrt{x}+\sqrt[5]{x^2}} \cdot \left(x^2 - \dfrac{2}{x}\right)\right]$
4. $y = \dfrac{(x^2-1)^3}{(4x^3-5)^2}$
5. $y = \sqrt{\dfrac{x^2+4x+2\sqrt{x}}{3x^2-7}}$
6. $y = \sqrt[3]{\dfrac{1+\cos x}{x\csc x^2}}$
7. $y = \sin\left[(4x^2+3x)^4\right]$
8. $y = x\tan\left(\dfrac{1}{x}\right) + \cot^2 4x$
9. $y = \dfrac{\cot(\cos^2 x)}{\sec^3 4x}$
10. $y = \dfrac{4x^2\left(\tan x^4 + 3\right)}{\sqrt{\sin\left[\csc\left(2x-3\right)\right]}}$
11. $y = \dfrac{\sqrt[3]{1-x} + \sec(x^3-2x)}{\pi^2 + 2x^2\cos 4x}$
12. $xy^3 = x + y^2 + 1$
13. $(x+y)^3 + (x-y)^3 = x^4 + y^4$
14. $x\cos y = (x-y^2)^2$
15. $\sin(xy) + \cot(x^2+y) = \sqrt{1000}$
16. $\sec(3xy^2) = y\csc x - \sqrt{x} + 2y$
17. $\dfrac{x^3}{4} - x\sqrt{y} = \sin\left[\tan(x^2+y^4)\right]$
18. $\cot\left(\dfrac{y}{x}\right) + \sin^2\left(x^3+y\right) = 3x^3$

III. Find $\dfrac{dm}{dn}$ if $nm^2 + 2m^3 = n - 2m$.

IV.
1. If $f(2) = 4$, $f'(2) = -3$, $g(2) = 2$ and $g'(2) = 6$, find $(f+g)'(2)$, $(f-g)'(2)$, $(f\cdot g)'(2)$ and $\left(\dfrac{g}{f}\right)'(2)$.
2. If $h(3) = 4$ and $h'(3) = -3$, find $\dfrac{d}{dx}\left[\dfrac{h(x)}{2x}\right]\Bigg|_{x=3}$
3. What is $(f\circ g)'(3)$ if $g(3) = 4$, $g'(3) = 2$, $f'(3) = -3$ and $f'(4) = -5$?

V.
1. Find the equation of the line normal to the graph of $y = x^3 - 3x + 1$ at the point where $x = 2$.
2. Find all points on the graph of $f(x) = 2\sin x + \sin^2 x$ at which the tangent line is horizontal.
3. Determine the equation of the line tangent to the graph of $y = x^2 - 1$ and parallel to the line $4x + y = 1$.
4. Given $f'(x) = \cot^2 x$ and $f\left(\dfrac{\pi}{3}\right) = \dfrac{\pi}{6}$.
   a) Find the equation of the line normal to the graph of f at $x = \dfrac{\pi}{3}$.
   b) Find $f'''(x)$.
5. Find the equation of the line tangent to the graph of $8(x^2+y^2)^2 = 100(x^2-y^2)$ at the point $(3,1)$.
6. At what point/s on the graph of $xy = (1-x-y)^2$ is the tangent line parallel to the y-axis?
7. Determine the equations of the two tangent lines to the graph of $y = 4x - x^2$ that passes through the point $(2,5)$.

VI.
1. Find all the derivatives of $f(x) = 10x^6 - 4x^5 + 3x^3 + x^2 - 12x + 21$.
2. Determine $y^{(4)}$ if $y = \dfrac{7}{x^4}$.
3. Evaluate $D_x^2\left(x\sqrt{9-x^2}\right)$ and $\dfrac{d^{99}}{dx^{99}}[\sin(2x)]$.

VII.
1. Find y'' at the point $(2,1)$ if $2x^2y - 4y^3 = 4$.
2. Given $x^2 + 4y^2 = 16$, show that $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4y^3}$.

VIII.
1. Determine if $f(x) = \begin{cases} \sqrt{x^2+4} & \text{if } x < 0 \\ [\![x]\!] + 2 & \text{if } 0 \le x \le 2 \\ \dfrac{x^2+x-9}{x-3} & \text{if } x > 2 \end{cases}$ is differentiable at $x = 0$ and $x = 2$.
2. Determine if $g(x) = \begin{cases} 4\cos x & \text{if } x < 0 \\ x - 5 & \text{if } 0 \le x < 4 \\ 4\sqrt{x} - 9 & \text{if } x \ge 4 \end{cases}$ is differentiable at $x = 0$ and $x = 4$.

IX.
1. Find real numbers a and b so that $g(x) = \begin{cases} ax^2 + 3bx & \text{if } x < 6 \\ \sqrt{5x-5} & \text{if } x \ge 6 \end{cases}$ is differentiable at $x = 6$.
2. Find real numbers a and b so that $g(x) = \begin{cases} ax^2 + \dfrac{b}{(x+1)^2} + a & \text{if } x \ge 1 \\ \sin(x-1) & \text{if } x < 1 \end{cases}$ is differentiable at $x = 1$.

X.
1. If $f(t) = at^2 + bt + c$, $f(1) = 5$, $f'(1) = 3$ and $f''(1) = -4$, what is $f(t)$?
2. If $f(x) + x\sin f(x) = x^2$, and $f(1) = 0$, find $f'(1)$.
3. Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = \dfrac{1}{1+x^2}$.