Math 54 Summer 2013
Fifth Long Exam

Show all necessary solutions and box all final answers. Use only black or blue ink.

I. Identify and sketch the domain of $f(x,y) = \sqrt{y-x} + \ln(y+x)$. (4 pts)

II. Show that the limit does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^6}{x^3 - x^2y^2 + y^6}$. (4 pts)

III. Consider the function

$$f(x,y) = \begin{cases} \dfrac{y^2 - x^2}{x^2y + y - x^3 - x}, & y \neq x \\ 0, & y = x \end{cases}.$$

Show that f has a removable discontinuity <u>at the point $(1,1)$</u>. Then redefine f to make it continuous <u>at the point $(1,1)$</u>. (4 pts)

IV. Given: $g(x,y,z) = e^{x^2y} - \sqrt{2xyz}$, where

$$x = 2s - t, \quad y = s^2t, \quad z = s^2 + t^2.$$

1. Find g_t at $(s,t) = (1,1)$ using chain rule. (4 pts)
2. Find $\dfrac{\partial^2 g}{\partial y \partial z}$ at $(s,t) = (1,1)$. (3 pts)

V. Find z_y at the point $(x,y,z) = (1,1,-1)$ given the equation (3 pts)

$$\tan(y-x) + \frac{xy}{z} = \overset{-2}{\cancel{1}} + y^{-2}z^2.$$

VI. Use linear approximation to estimate the value of $(0.98)^5 + (3.01)^2$. (4 pts)

VII. The volume of a pyramid with height h and a square base with side b is given by $V = \frac{1}{3}b^2h$. Use differentials to approximate the change in volume of the pyramid if h is decreased from 3m to 2.97m and b is increased from 1m to 1.1m. (4 pts)

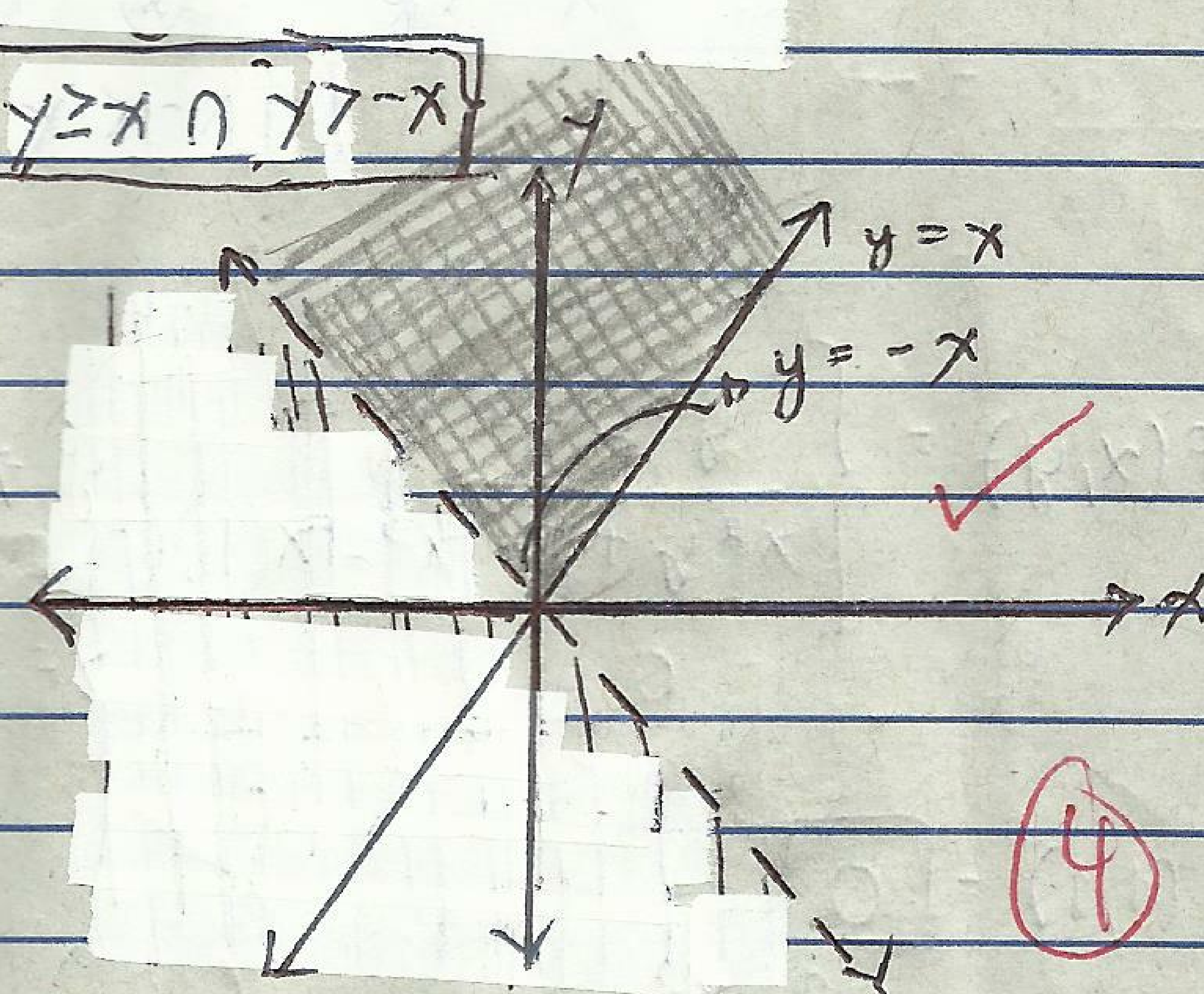
END OF EXAM —
total: 30 points

I.

$f(x,y) = \sqrt{y-x} + \ln(y+x)$

①	②
dom $\sqrt{y-x}$	dom $\ln(y+x)$
$y-x \geq 0$	$y+x > 0$
$y \geq x$	$y > -x$

$\boxed{\text{dom } f : \{(x,y) \in \mathbb{R}^2 : y \geq x \cap y > -x\}}$



4

II. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^6}{x^3-x^2y^2+y^6}$ DNE

$\because$

* Along $y=x$

$$\lim_{x\to 0} \frac{x^3+x^6}{x^3-x^4+x^6} = \lim_{x\to 0} \frac{\cancel{x^3}(1+x^6)}{\cancel{x^3}(1-x+x^3)} = \boxed{1}$$

* Along $y=x^2$

$$\lim_{x\to 0} \frac{x^3+(x^2)^6}{x^3-x^2(x^2)^2+(x^2)^6} = \lim_{x\to 0} \frac{x^3+x^{12}}{x^3-x^6+x^{12}} = \lim_{x\to 0} \frac{\cancel{x^3}(1+x^9)}{\cancel{x^3}(1-x^3+x^9)}$$

$$= \boxed{1}$$

* Along $y=x^{1/2}$

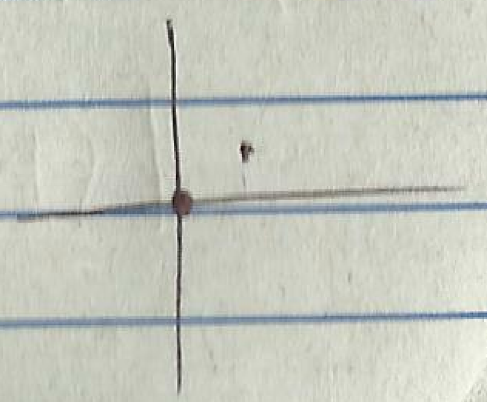
$$\lim_{x\to 0} \frac{x^3+(x^{1/2})^6}{x^3-x^2(x^{1/2})^2+(x^{1/2})^6} = \lim_{x\to 0} \frac{2x^3}{x^3-x^3+x^3}$$

4

$$\lim_{x\to 0} \frac{2\cancel{x^3}}{\cancel{x^3}} = \lim_{x\to 0} 2 = \boxed{2}$$

Since $2 \neq 1$, (for 2 distinct curves)
Limit DNE.

III. $f(x,y) = \begin{cases} \dfrac{y^2 - x^2}{x^2 y + y - x^3 - x} & y \neq x \\ 0 & y = x \end{cases}$



- $f(1,1) = \boxed{0}$
- $\lim\limits_{(x,y)\to(1,1)} \dfrac{y^2 - x^2}{x^2 y + y - x^3 - x} \quad \left(\dfrac{0}{0}\right)$ discontinuous at $(1,1)$

$$\lim_{(x,y)\to(1,1)} \frac{(y-x)(y+x)}{\underbrace{(x^2 y + y) - (x^3 + x)}}$$

$$\frac{(y-x)(y+x)}{y(x^2+1) - x(x^2+1)} = \frac{\cancel{(y-x)}(y+x)}{\cancel{(y-x)}(x^2+1)}$$

$$\lim_{(x,y)\to(1,1)} \frac{x+y}{x^2+1} = \frac{1+1}{1+1} = \boxed{1}$$

- $f(1,1) \neq \lim\limits_{(x,y)\to(1,1)} f(x,y)$ There is a removable discontinuity at $(1,1)$

Redefining f:

$f(1,1) = 1$ to make the function continuous at that point.

ok

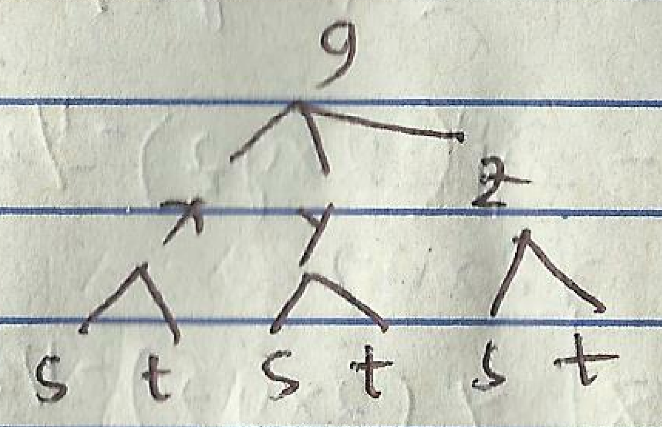
4

IV. $g(x,y,z) = e^{x^2 y} - \sqrt{2xyz}$

$x = 2s - t$

$y = s^2 t$

$z = s^2 + t^2$



1) $g_t = g_x \cdot x_t + g_y \cdot y_t + g_z \cdot z_t$

$g_x = e^{x^2 y}(2xy) - \frac{1}{2}(2xyz)^{-1/2}(2yz)$

$x_t = -1$

$g_y = e^{x^2 y}(x^2) - \frac{1}{2}(2xyz)^{-1/2}(2xz)$

$y_t = s^2$

$g_z = -\frac{1}{2}(2xyz)^{-1/2}(2xy)$

$z_t = 2t$

at $(1,1)$ st

$x = 1$

$y = 1$

$z = 2$

$g_x = e(2) - \frac{1}{2}(4)^{-1/2}(4)$

$g_x = 2e - 1$

$g_y = e(1) - \frac{1}{2}(4)^{-1/2}(4)$

$g_y = e - 1$

$g_z = -\frac{1}{2}(4)^{-1/2}(2)$

$g_z = \frac{-1}{\sqrt{4}} = \frac{-1}{2}$

$g_t = (2e-1)(-1) + (e-1)(1) + (-\frac{1}{2})(2)$

$= \cancel{1} - 2e + e \cancel{-1} - 1$

$= \boxed{-e-1}$

4

$$\frac{\partial^2 g}{\partial y \partial z}$$

$x = 1$
$y = 1$
$z = 2$

$$\frac{\partial g}{\partial z} = 0 - \frac{1}{2}(2xyz)^{-1/2}(2xy)$$

$$= -xy(2xyz)^{-1/2}$$

$$\frac{\partial^2 g}{\partial y \partial z} = -x\left[y(2xyz)^{-1/2}\right]$$

$$= -x\left[(2xyz)^{-1/2} + y(-1/2)(2xyz)^{-3/2}(2xz)\right]$$

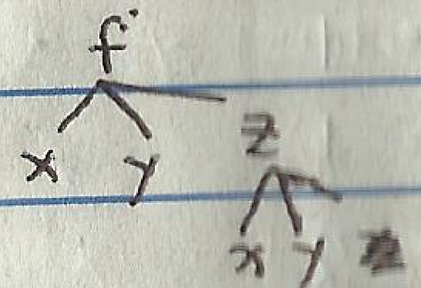
$$= -1\left[(4)^{-1/2} + (1)(-1/2)(4)^{-3/2}(4)\right]$$

$$= -1\left[\frac{1}{2} + \left(-\frac{1}{4}\right)\right]$$

$$= \boxed{-1/4}$$

3

V. z_y at $(x,y,z) = (1,1,-1)$ $\quad z_y = \frac{\partial z}{\partial y}$



$$\tan(y-x) + \frac{xy}{z} = -2 + y^{-2}z^{2}$$

$$\tan(y-x) + \frac{xy}{z} - y^{-2}z^{2} + 2 = 0$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-2}{1}$$

answer: $\boxed{-2}$

F_y : $\sec^2(y-x)(1) + \frac{x}{z} - z^2(-2)(y^{-3})$

$F_y(1,1,-1)$: $\cancel{1 + (-1)} - (-1)^3(-2)(1)$

: 2

F_z : $0 + -\frac{xy}{z^2} - y^{-2}(2)(z)$

$F_z(1,1,-1)$: $-1 - (1)(2)(-1) = 1$

3

VI.

$$(0.98)^5 + (3.01)^2 \approx 10$$

$f(x,y) = x^5 + y^2$ $\quad$ Let $x_0 = 1$, $y_0 = 3$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$L(1,3) = f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3)$$

$f(1,3) = 10$

$$L(1,3) = 10 + 5(x-1) + 6(y-3)$$
$$= 10 + 5(0.98-1) + 6(3.01-3$$

$f_x = 5x^4$

$f_x(1,3) = 5$

$$= 10 + 5(-0.02) + 6(0.01)$$
$$= 10 - 0.10 + 0.06$$

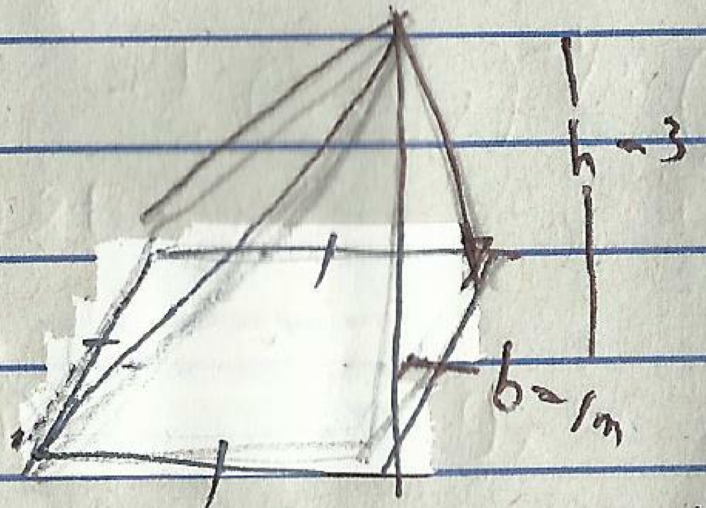
$f_y = 2y$

$f_y(1,3) = 6$

$$\boxed{L(1,3) = 9.96}$$

4

VII.



$V = \frac{1}{3}b^2h$ $\quad dv \approx \Delta v$

total differential $= \frac{\partial V}{\partial b} \cdot db + \frac{\partial v}{\partial h} \cdot$

$$dv = \frac{2}{3}bh(db) + \frac{1}{3}b^2(dh)$$

$db = 0.1$ m

$dh = -0.03$ m

$$dv = \frac{2}{3}(1)(3)(0.1) + \frac{1}{3}(1)^2(-0.03$$

$$dv = 0.2 - 0.01$$

$$dv = \boxed{0.19 \text{ m}^3}$$

4