I. A. Find the following antiderivatives.

1. $\int x\csc^2 x\,dx$
2. $\int z\cosh^{-1} z\,dz$
3. $\int \cos 4x\cos 3x\,dx$
4. $\int \cos 7\alpha\,\sin 3\alpha\,d\alpha$
5. $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$
6. $\int \sin^2(4y+1)\cos^2(4y+1)\,dy$
7. $\int \frac{3x^2-1}{2x\sqrt{x}}\tan^{-1} x\,dx$
8. $\int \frac{y^3+2y^2+2}{y\,(y^2+1)^2}\,dy$
9. $\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$
10. $\int \left(\sin^2 3\theta + \frac{\sin^3\theta}{\sqrt{\cos\theta}}\right)\,d\theta$
11. $\int \ln(x^2+1)\,dx$
12. $\int \frac{w^2-6w+3}{w^2(w^2+3)}\,dw$
13. $\int \frac{1}{\sqrt{x^2+2x-15}}\,dx$
14. $\int e^w\sec^4 e^w\tan^4 e^w\,dw$
15. $\int \frac{1}{x^3(x-1)}\,dx$
16. $\int \sinh z\ln^2(\cosh z)\,dz$
17. $\int \frac{1}{(16+x^2)^2}\,dx$
18. $\int \frac{\csc^6(\log_5 t)\cot^2(\log_5 t)}{t}\,dt$
19. $\int \sin(\ln x)\,dx$
20. $\int \frac{z^3+2z^2+3z-2}{(z^2+2z+2)^2}\,dz$
21. $\int \frac{\tan^3(\ln x)\sec^8(\ln x)}{x}\,dx$
22. $\int \frac{\sqrt{2m-m^2}}{m}\,dm$
23. $\int e^{2x}\ln(e^x+1)\,dx$
24. $\int (\sqrt{\sin\theta}+\cos\theta)^2 d\theta$
25. $\int \frac{\tan^4(\cosh^{-1} 2x)}{\sqrt{4x^2-1}}\,dx$
26. $\int \frac{\sin 4\theta\cos 3\theta}{\cos 2\theta}\,d\theta$
27. $\int \sqrt{16-e^{2x}}\,e^x\,dx$
28. $\int \frac{y\csc^3(\tanh y^2)}{\cosh^2 y^2}\,dy$
29. $\int \frac{4x-3}{x^2-2x-3}\,dx$
30. $\int \frac{\tan^5(\sinh^{-1} y)\sqrt{\sec(\sinh^{-1} y)}}{\sqrt{y^2+1}}\,dy$
31. $\int \frac{\csc^4 x}{\cot^2 x}\,dx$
32. $\int \frac{45-4z}{(z^2+2)(3-z)^2}\,dz$
33. $\int \sqrt{9+\cos^2 x}\,\sin x\,dx$
34. $\int e^{e^v+v}\cos e^v\,dv$
35. $\int \frac{x^3-2x}{(x^2+1)^2}\,dx$
36. $\int \frac{2^t}{(2^t-2)(4^t+1)}\,dt$

B. Evaluate the following integrals.

1. $\int_0^{\frac{\pi}{2}} \cos^3\theta\sqrt[5]{\sin\theta}\,d\theta$
2. $\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$
3. $\int_0^1 \frac{w}{w^2+4w+13}\,dw$
4. $\int_0^1 \sin^2(\pi x)\cos^2(\pi x)\,dx$
5. $\int_0^{\frac{\pi}{2}} e^x\sin x\,dx$
6. $\int_{-1}^0 \coth t\,dt$
7. $\int_{-\infty}^0 \frac{1}{x^2+16}\,dx$
8. $\int_1^{+\infty} \frac{1}{v\sqrt{4v^2-1}}\,dv$

9. $\displaystyle\int_{-\infty}^{\infty} \frac{1+x^2}{1-x^2}\,dx$

10. $\displaystyle\int_{0}^{1} \frac{1}{e^{-x}-e^{x}}\,dx$

11. $\displaystyle\int_{-\infty}^{4} (s^2-6s+10)^{-\frac{5}{2}}\,ds$

12. $\displaystyle\int_{0}^{2} \frac{1}{(2x-1)^{2/3}}\,dx$

13. $\displaystyle\int_{0}^{+\infty} r\,e^{r^2}\,dr$

14. $\displaystyle\int_{0}^{2} \frac{3x+2}{(x-2)(x^2+4)}\,dx$

II. 1. Let f be a function continuous on $[1,4]$ whose graph is tangent to the x-axis at $x=1$. If $f(4)=3$ and $f'(4)=2$, determine the value of $\displaystyle\int_1^4 xf''(x)dx$.

2. Show that $\displaystyle\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = 0$ for positive integers m and n with $m \neq n$.

3. Show that $\displaystyle\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+C$, $|x|>a>0$ using

a. trigonometric substitution;

b. partial fractions.

4. Show that $\displaystyle\int_2^{+\infty} \frac{1}{x(\ln x)^p}\,dx$ is convergent if $p>1$ and is divergent if $p \leq 1$.

III. A. Solve the following differential equations.

1. $3dy + yx^4dx = 2y^3dy$
2. $\dfrac{dy}{dx} = \dfrac{xe^x}{y\sqrt{1+y^2}}$
3. $\dfrac{dy}{dx} = \dfrac{e^y \sin^2 x}{y \sec x}$
4. $\dfrac{d^2y}{dx^2} = \dfrac{x}{(4x^2+1)^2}$
5. $\dfrac{d^2y}{dx^2} = \dfrac{1}{x\sqrt{x^2-1}}$
6. $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^2}$, $y(0)=0$
7. $xy'+y=y^2$, $y(1)=-1$

B. Solve the following problems completely.

1. Find the equation of the curve that passes through the point $Q(0,1)$ such that the slope of the tangent line to the curve at $P(x,y)$ is xy.
2. The points $Q_1(1,3)$ and $Q_2(0,2)$ are on a curve, and at any point $P(x,y)$ on the curve, $\dfrac{d^2y}{dx^2} = 2-4x$. Find the equation of the curve.
3. Find the function f such that $f'(x)=f(x)(1-f(x))$ and $f(0)=\frac{1}{2}$.
4. The transport of a substance across a capillary wall in lung physiology has been modeled by the differential equation

$$\frac{dh}{dt} = -\frac{R}{V}\left(\frac{h}{k+h}\right),$$

where h is the hormone concentration in the bloodstream, t is time, R is the maximum transport rate, V is the volume of the capillary, and k is a positive constant that measures the affinity between the hormones and the enzymes that assist the process. Solve this differential equation to find a relationship between h and t.
5. Find the orthogonal trajectories of the family of curves.

a. $x^2+2y^2=K$ b. $y^2=Kx^3$ c. $y=\dfrac{x}{1+Kx}$

6. A bacteria culture starts with 500 bacteria. After 3 hours, there are 8000 bacteria. Find:

a. the initial population;

b. the formula for the population after t hours;

c. the number of bacteria after 5 hours;

d. the rate of growth after 5 hours;

e. the time when the population reaches 200000.

7. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.
   a. Find the mass that remains after t years.
   b. How much of the sample remains after 100 years?
   c. After how long will only 1 mg remain?
8. After 3 days, a sample of Radon-222 decays to 58% of its original amount. Find:
   a. the half-life of Radon-222;
   b. the time it would take for the sample to decay to 10% of its original amount.