Math 22 **2nd Semester AY 2018-2019**

1st Long Exam **11 February 2019**

This exam is for 90 minutes only. Use blue or black non-erasable ink only. Show neat and complete solutions, and box all final answers. The use of electronic devices is not allowed during the exam.

Any form of cheating in examinations or any act of dishonesty in relation to studies, such as plagiarism, shall be subject to disciplinary action.

I. Evaluate the following integrals.

1. $\displaystyle\int x^2 e^{-2x}\, dx$ (4 pts)

2. $\displaystyle\int \csc^7 x \cot^5 x\, dx$ (5 pts)

3. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\, dx$ (5 pts)

II. Determine if each improper integral is convergent or divergent.

1. $\displaystyle\int_0^{\pi} \frac{\sin x}{(\cos x)^{\frac{4}{3}}}\, dx$ (5 pts)

2. $\displaystyle\int_2^{\infty} \frac{x^2-12}{x^4+4x^2}\, dx$ (7 pts)

III. Let C be the parametric curve given by

$$\begin{aligned} x &= t^2-5t+6 \\ y &= (2t-2)^{\frac{5}{2}} \end{aligned}, \; t \in \mathbb{R}.$$

1. Find an equation of the tangent line to C at the point where $t=\frac{3}{2}$. (3 pts)

2. Determine if C is concave up or concave down at the point where $t=\frac{3}{2}$. (3 pts)

3. **Set-up** a definite integral equal to the arc length of the portion of C from the point $(2,0)$ to the point $(0,32)$. (3 pts)

END OF EXAM

TOTAL: 35 points

MATH 21: Elementary Analysis II
First Exam

Second Semester AY 2018-2019
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I. Evaluate the following integrals.

① $\int x^2 e^{-2x}\,dx$

Soln 1: Let $u = x^2$, $du = 2x\,dx$; $v = -\frac{1}{2}e^{-2x}$, $dv = e^{-2x}\,dx$

$$= -\frac{1}{2}x^2e^{-2x} + \cancel{2}\cdot\frac{1}{\cancel{2}}\int xe^{-2x}\,dx$$

Let $\bar{u} = x$, $d\bar{u} = dx$; $\bar{v} = -\frac{1}{2}e^{-2x}$, $d\bar{v} = e^{-2x}\,dx$

$$= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$$

$$= \boxed{-\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C}$$

Soln 2:

u		dv
x^2	$(+)$	e^{-2x}
$2x$	$(-)$	$-\frac{1}{2}e^{-2x}$
2	$(+)$	$\frac{1}{4}e^{-2x}$
0		$-\frac{1}{8}e^{-2x}$

$$= \boxed{-\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C}$$

② $\int \csc^7 x\cot^5 x\,dx$

$$= \int \csc^6 x\cot^4 x\cdot\csc x\cot x\,dx$$
$$= \int \csc^6 x\,(\cot^2 x)^2\csc x\cot x\,dx$$
$$= \int \csc^6 x\,(\csc^2 x - 1)^2\csc x\cot x\,dx$$

Let $t = \csc x$ $\quad dt = -\csc x\cot x\,dx$

$$= -\int t^6(t^2-1)^2\,dt$$
$$= -\int t^6(t^4 - 2t^2 + 1)\,dt$$
$$= -\int (t^{10} - 2t^8 + t^6)\,dt$$
$$= -\left[\frac{1}{11}t^{11} - \frac{2}{9}t^9 + \frac{1}{7}t^7\right] + C$$
$$= \boxed{-\left[\frac{1}{11}\csc^{11}x - \frac{2}{9}\csc^9 x + \frac{1}{7}\csc^7 x\right] + C}$$

③ $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$

$$= \int_0^1 \frac{x^2}{\sqrt{(2)^2-x^2}}\,dx$$

Let $x = 2\sin\theta \Rightarrow \cos\theta = \frac{\sqrt{4-x^2}}{2}$; $dx = 2\cos\theta\,d\theta$

$$= \int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta$$
$$= \int \frac{4\sin^2\theta}{\cancel{2\cos\theta}}\cdot\cancel{2\cos\theta}\,d\theta$$
$$= 4\int \sin^2\theta\,d\theta$$
$$= 4\int \frac{1-\cos 2\theta}{2}\,d\theta$$
$$= 2\int (1-\cos 2\theta)\,d\theta$$
$$= 2\cdot\left[\theta - \frac{1}{2}\sin 2\theta\right]$$
$$= 2\theta - \sin 2\theta$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\sin\theta\cos\theta$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \cancel{2}\cdot\frac{x}{\cancel{2}}\cdot\frac{\sqrt{4-x^2}}{2}$$
$$= \left(2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2}\right)\Big|_0^1$$
$$= \left(2\sin^{-1}\left(\frac{1}{2}\right) - \frac{\sqrt{3}}{2}\right) - (0-0)$$
$$= 2\cdot\frac{\pi}{6} - \frac{\sqrt{3}}{2}$$
$$= \boxed{\frac{\pi}{3} - \frac{\sqrt{3}}{2}}$$

II. Determine if each improper integral is convergent or divergent.

① $\int_0^\pi \frac{\sin x}{(\cos x)^{4/3}}\,dx$

domain $= \mathbb{R} \setminus \{$odd multiples of $\frac{\pi}{2}\}$

$$= \int_0^{\pi/2} \frac{\sin x}{(\cos x)^{4/3}}\,dx + \int_{\pi/2}^{\pi} \frac{\sin x}{(\cos x)^{4/3}}\,dx$$
$$= \lim_{a\to\frac{\pi}{2}^-}\int_0^a \frac{\sin x}{(\cos x)^{4/3}}\,dx + \lim_{b\to\frac{\pi}{2}^+}\int_b^\pi \frac{\sin x}{(\cos x)^{4/3}}\,dx.$$

Now, $\int \frac{\sin x}{(\cos x)^{4/3}}\,dx$

Let $t = \cos x$ $\quad dt = -\sin x\,dx$

$$= -\int t^{-4/3}\,dt$$
$$= -\frac{t^{-1/3}}{-1/3}$$
$$= \frac{3}{\sqrt[3]{t}}$$
$$= \frac{3}{\sqrt[3]{\cos x}}$$

Hence, $\lim_{a\to\frac{\pi}{2}^-}\left[\frac{3}{\sqrt[3]{\cos a}} - \frac{3}{\sqrt[3]{\cos 0}}\right]$

$$+ \lim_{b\to\frac{\pi}{2}^+}\left[\frac{3}{\sqrt[3]{\cos\pi}} - \frac{3}{\sqrt[3]{\cos b}}\right]$$
$$= \lim_{a\to\frac{\pi}{2}^-}\frac{3}{\sqrt[3]{\cos a}} - 3 + (-3) - \lim_{b\to\frac{\pi}{2}^+}\frac{3}{\sqrt[3]{\cos b}}$$
$$= \frac{3}{\sqrt[3]{0^+}} - 6 - \frac{3}{\sqrt[3]{0^-}}$$
$$= +\infty - 6 + \infty$$
$$= \boxed{+\infty \;\therefore \text{divergent}}$$

② $\int_2^\infty \frac{x^2-12}{x^4+4x^2}\,dx$

$$\frac{x^2-12}{x^4+4x^2} = \frac{x^2-12}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$$
$$= \frac{Ax(x^2+4) + B(x^2+4) + (Cx+D)x^2}{x^2(x^2+4)}$$

if $x=0$: $0^2-12 = A\cdot0(0+4) + B(0^2+4) + (C\cdot0+D)\cdot0^2$

$-12 = 0 + 4B + 0$

$B = -3$

if $x=1$: $1-12 = A\cdot1(1+4) - 3(1+4) + (C\cdot1+D)\cdot1$

$-11 = 5A - 15 + C + D$

$4 = 5A + C + D$ (Eqtn 1)

if $x=-1$: $1-12 = -A(1+4) - 3(1+4) + (C(-1)+D)\cdot(-1)^2$

$-11 = -5A - 15 - C + D$

$4 = -5A - C + D$ (Eqtn 2)

Adding Equations 1 & 2,

$$+\begin{cases}4 = 5\cancel{A} + \cancel{C} + D\\ 4 = -5\cancel{A} - \cancel{C} + D\end{cases}$$
$$8 = 2D \Rightarrow D = 4$$

Going back to eqtn 1, w/ $D=4$,

$$\begin{cases}4 = 5A+C+D\\ 4 = -5A-C-D\end{cases} \Leftrightarrow \begin{cases}5A+C = 0\\ -5A-C = 0\end{cases}$$
$$\Rightarrow 5A = -C$$

if $x=2$: $4-12 = A\cdot2(8) - 3(8) + ((-5A)\cdot2 + 4)(4)$

$\cancel{-8} = 16A - \cancel{24} - 40A + \cancel{16}$

$0 = -24A \to A = 0$

$\Rightarrow C = 0$

$$\therefore \int_2^\infty \frac{x^2-12}{x^4+4x^2}\,dx = \int_2^\infty \frac{-3}{x^2}\,dx + \int_2^\infty \frac{4}{x^2+4}\,dx$$
$$= \lim_{t\to\infty}\frac{3}{x}\Big|_2^t + \lim_{s\to\infty}4\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_2^s$$
$$= \left(\frac{3}{\infty} - \frac{3}{2}\right) + \left(2\tan^{-1}(\infty) - 2\tan^{-1}(1)\right)$$
$$= \left(0 - \frac{3}{2}\right) + \left(2\cdot\frac{\pi}{2} - 2\cdot\frac{\pi}{4}\right)$$
$$= -\frac{3}{2} + \pi - \frac{\pi}{2}$$
$$= \frac{\pi}{2} - \frac{3}{2} = \boxed{\frac{\pi-3}{2}}$$

Math 22: Elementary Analysis II
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III. Let C be the parametric curve given by
$x = t^2 - 5t + 6$; $y = (2t-2)^{5/2}$; where $t \in \mathbb{R}$.

① Find an equation of the tangent line to C at the point where $t = \frac{3}{2}$.

$\frac{dy}{dt} = \frac{5}{2}(2t-2)^{3/2} \cdot 2$; $\frac{dx}{dt} = 2t - 5$

$\left.\frac{dy}{dt}\right|_{t=3/2} = \frac{5}{\cancel{2}}\left(\cancel{2}\cdot\frac{3}{\cancel{2}}-2\right)^{3/2}\cdot\cancel{2} = 5$; $\left.\frac{dx}{dt}\right|_{t=3/2} = \cancel{2}\cdot\frac{3}{\cancel{2}} - 5 = -2$

$\Rightarrow \frac{dy}{dx} = -\frac{5}{2}$

Also, at $t = \frac{3}{2}$, $x = \left(\frac{3}{2}-3\right)\left(\frac{3}{2}-2\right) = -\frac{3}{2}\cdot\frac{-1}{2} = \frac{3}{4}$

$y = \left(\cancel{2}\frac{3}{\cancel{2}}-2\right)^{5/2} = 1$

$\therefore$ eqtn of TL is $\boxed{y - 1 = -\frac{5}{2}\left(x - \frac{3}{4}\right)}$

② Determine if C is concave up or concave down at the point where $t = \frac{3}{2}$.

Since $\frac{dy}{dx} = \frac{5(2t-2)^{3/2}}{2t-5}$, this implies that

$$\frac{d[dy/dx]}{dt} = \frac{(2t-5)\frac{3}{\cancel{2}}\cdot 5(2t-2)^{1/2}\cdot\cancel{2} - 5(2t-2)^{3/2}\cdot 2}{(2t-5)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(2t-5)\cdot 15(2t-2)^{1/2} - 10(2t-2)^{3/2}}{(2t-5)^3}$$

At $t = \frac{3}{2}$, $\left.\frac{d^2y}{dx^2}\right|_{t=\frac{3}{2}} = \frac{\left(\cancel{2}\cdot\frac{3}{\cancel{2}}-5\right)\cdot 15\left(\cancel{2}\cdot\frac{3}{\cancel{2}}-2\right)^{1/2} - 10\left(\cancel{2}\cdot\frac{3}{\cancel{2}}-2\right)^{3/2}}{\left(\cancel{2}\cdot\frac{3}{\cancel{2}}-5\right)^3}$

$= \frac{(-2)\cdot(15) - 10}{(-2)^3}$

$= \frac{-40}{-8} = 5 > 0$ $\therefore$ $\boxed{\text{C is concave up at } t = \frac{3}{2}}$

③ Set up the integral equal to the arc length of the portion of C from the point (2,0) to the point (0,32).

if $(x,y) = (2,0)$, then $t^2 - 5t + 6 = 2$ and $(2t-2)^{5/2} = 0$
$t^2 - 5t + 4 = 0$; $2t - 2 = 0$
$(t-4)(t-1) = 0$; $t = 1$
$t = 4, t = 1$

$\Rightarrow$ if $(x,y) = (2,0)$, $t = 1$

if $(x,y) = (0,32)$, then $t^2 - 5t + 6 = 0$ and $(2t-2)^{5/2} = 32$
$(t-3)(t-2) = 0$; $2t - 2 = 4$
$t = 3, t = 2$; $t = 3$

$\Rightarrow$ if $(x,y) = (0,32)$, $t = 3$

$\therefore$ $\boxed{L = \int_1^3 \sqrt{(2t-5)^2 + \left(5(2t-2)^{3/2}\right)^2}\, dt}$