Exercises on the Mean Value Theorem, Relative Extrema, Increasing and Decreasing Functions, Points of Inflection, Concavity, Absolute Extrema

I. Determine whether each of the following statements is true or false.

1. If f has a point of inflection at $P(c, f(c))$ and $f''(c)$ exists, then $f''(c) = 0$.
2. If f is increasing on (a, c) and decreasing on (c, b), then f does not have a relative minimum at $x = c$.
3. If f has an absolute maximum on an interval at $x = c$, then f has a relative maximum at $x = c$.
4. If f' has a relative minimum at $x = c$, then either $f''(c) = 0$ or $f''(c)$ does not exist.
5. The graph of a function f cannot have three horizontal asymptotes.
6. A point of inflection cannot be a relative extremum point at the same time.
7. If $f'(c) = 0$ and f has a relative maximum at $x = c$, then $f''(c) < 0$.
8. If $f(a) = f(b) = 0$ and f is continuous on $[a, b]$ but f is not differentiable on (a, b), then there is no $c \in (a, b)$ such that $f'(c) = 0$.
9. If f is increasing on (a, b), then $f'(x) > 0$ on (a, b).

II. Sketch the graphs of the functions below following the suggested steps given in class.

1. $f(x) = x^3 - 3x^2 + 2$
2. $f(x) = (x-2)(x+2)^3$
3. $f(x) = x^{\frac{2}{3}} - 2x^{-\frac{1}{3}}$
4. $f(x) = \dfrac{\sqrt{x}(x-5)^2}{4}$
5. $f(x) = \dfrac{x}{(x+1)^2}$
6. $f(x) = \dfrac{-x^2}{(x+2)^2}$
7. $f(x) = \dfrac{x^2-2}{x-1}$
8. $f(x) = \dfrac{(x-1)^3}{x^2}$
9. $f(x) = \dfrac{x^3}{x^2+1}$
10. $f(x) = \dfrac{8x^2+36x+36}{(x+2)^2}$
11. $f(x) = \dfrac{2x^3}{x^2-4}$
12. $f(x) = \dfrac{x^3-1}{x^2-1}$

III. Sketch a possible graph of the function f having the following characteristics.

(i) f is continuous everywhere except at $x = 3$

(ii) $f(-4) = -3, f(0) = 0, f(3) = 2$

(iii) $\lim\limits_{x\to-\infty} f(x) = -1,\ \lim\limits_{x\to3^-} f(x) = +\infty,\ \lim\limits_{x\to3^+} f(x) = -2$

(iv) $f'(-4)$ does not exist, $f'(0) = 0$, $f'(x) < 0$ when $x < -4$, $f'(x) > 0$ when $-4 < x < 3$, $f'(x) < 0$ when $x > 3$

(v) $f''(x) < 0$ when $x < -4$, $f''(x) < 0$ when $-4 < x < 0$, $f''(x) > 0$ when $0 < x < 3$, $f''(x) < 0$ when $x > 3$

IV.

1. If $f(x) = ax^3 + bx^2 + cx + d$, solve for a, b, c and d so that f will have a relative extremum at $(0, 3)$ and a point of inflection at $(1, -1)$.
2. Show that any polynomial function of degree three has exactly one point of inflection.
3. Suppose $g(x) = x^r - rx + 1$, where $r \in \mathbb{Q}$, $r > 0$ and $r \neq 1$. Show that
   i. if $0 < r < 1$, then g has a relative maximum at 1.
   ii. if $r > 1$, then g has a relative minimum at 1.

V. Find absolute extremum values of the following functions on the given intervals, if any, and determine the values of x where they are attained.

1. $f(x) = 4 - 3x$, $(-1, 2]$
2. $f(x) = x^2 - 2x + 4$, $(-\infty, +\infty)$
3. $f(x) = 2\cos x$, $(-\frac{2\pi}{3}, \frac{\pi}{3})$
4. $f(x) = \sqrt{4 - x^2}$, $(-2, 2)$
5. $f(x) = \dfrac{16}{x} - x^2$, $(-\infty, -1]$
6. $f(x) = |x - 4| + 1$, $(0, 6)$

VI.

1. Verify that the following functions satisfy the conditions of Rolle's Theorem on the given interval and find all numbers that satisfy the conclusion of the theorem.

   i. $x^2 - 4x + 2$, [0,4] ii. $\sin 2x$, $[0, \frac{\pi}{2}]$ iii. $x\sqrt{x+6}$, [-6,0]

2. Verify that the following functions satisfy the hypothesis of the Mean Value Theorem on the given interval and find all numbers that satisfy the conclusion of the theorem.

   i. $6x^2 + x - 2$, [-1,1] ii. $\dfrac{x}{x+2}$, [1,4] iii. $x^{\frac{2}{3}}$, $[0, 2]$

VII.

1. Is it possible for a function f to have all of the following properties: $f(0) = 0$, $f(2) = 2010$ and $f'(x) \leq 1000$ for all x?
2. Suppose f is differentiable everywhere and $f'(x) \leq 10$ for all x. If $f(-3) = 2$, determine whether $f(2)$ has a maximum possible value or a minimum possible value and find that value.
3. Suppose g is continuous on $[a, b]$ and $g''(x)$ exists $\forall x \in (a, b)$. Show that if there are three distinct numbers in $[a, b]$ for which $g(x) = 2$, then there is at least one c in (a, b) such that $g''(c) = 0$.
4. Show that there is at least one value of x in $(-1, 1)$ for which $24x^3 - 3x^2 - 36x + 1$ is zero.
5. Show that for any a, b such that $a \neq b$, $|\sin b - \sin a| \leq |b - a|$.