

I. TRUE or FALSE. Write “True” if the statement is always true, otherwise, write “False”.
Let A, B and C be sets.

- The field is closed under the operation of set difference.
- Some subsets of the set of real numbers are not Borel sets.
- If $A - (B \cap C) = \emptyset$ then $A \subset (B \cap C)$.
- If $A \cap B \cap C = \emptyset$ then A, B and C are pairwise disjoint.
- If $\{A, B - A, C - (A \cup B)\}$ is a partition of $A \cup B \cup C$.
- \mathcal{C} is closed under the intersection of two sets if and only if it is closed under finite intersection.
- If $A \subset B$ then $C = \{\emptyset, \Omega, A, A^c, B, B^c, B - A, (B - A)^c\}$ is a field.
- $\mathcal{C} = \{\emptyset, \Omega, A, A^c\}$ is a sigma-field.
- A sigma-field is closed under generalized union.
- The set of all positive rational numbers is a Borel set.

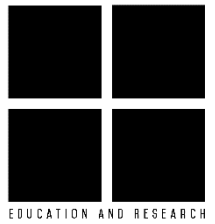
II. Let $\Omega =$ set of reals. For the following sequences of $\{A_n\}$

- Write ‘M’ if it is a monotone sequence; otherwise, write ‘N’
- If it is a monotone sequence, identify the limit of the sequence. If it is not a monotone sequence, identify the countable union and countable intersection.

- $A_n = \{0, 1, 2, \dots, n\}$
- $A_n = \{5 - 1/n, 8 - 1/n\}$
- $A_n = \{-2/n, 2/n\}$
- $A_n = \{5 - 2n, 8 + 2n\}$
- $A_n = \{n - 1, n, n + 1\}$

III. State the complete definition of the following terms by filling in the blanks.

- The class \mathcal{F} is a field iff _____.
- The class $\mathcal{F}(\mathcal{C})$ is the minima sigma field containing the class \mathcal{C} iff _____.
- The Borel field, denoted by \mathcal{B} is _____ where $\Omega =$ _____ and $\mathcal{C} =$ _____.
- The class $\{A_1, A_2, \dots, A_n\}$ is pairwise disjoint iff _____.
- $\omega \in \bigcup_{\lambda \in \Delta} A_\lambda$ iff _____.
- The class \mathcal{C} is closed under finite union iff _____.
- The class \mathcal{C} is closed under countable intersection iff _____.



IV. Define $\Omega = \{a, b, c, d, e, f\}$
 $\mathcal{C} = \{\{a\}, \{a, c\}, \{a, d, e\}\}$

1. Construct the minimal sigma-field containing \mathcal{C} .
2. Specify by roster method a partition of Ω .

V. A. Suppose a newly-married couple plans to have four children.

Define $\Omega = \{(\omega_1, \omega_2, \omega_3, \omega_4) : \omega_i = \text{sex of the } i\text{th child} \in \{G, B\}, i = 1, 2, 3, 4\}$

$G_i = \text{set of elements of } \Omega \text{ where the } i\text{th child is a girl, } i = 1, 2, 3, 4$

Express the following sets as a composition of the G_i 's

1. $A = \text{the set containing elements of } \Omega \text{ where only the second child is a boy}$
2. $B = \text{the set containing elements of } \Omega \text{ where there is at least one girl}$
3. $C = \text{the set containing elements of } \Omega \text{ where there is exactly 1 boy}$

B. Suppose a die is tossed four times.

Define $\Omega = \{(\omega_1, \omega_2, \omega_3, \omega_4) : \omega_i = \text{no. of dots on the } i\text{th toss} \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2, 3, 4\}$

$A_j = \text{set of elements of } \Omega \text{ where the } j \text{ dots come up on the } i\text{th toss, } i=1,2,3,4 \text{ and } j=1,2,3,4,5,6$

Express the following sets as a composition of A_j

1. $A = \text{the set containing elements of } \Omega \text{ where 3 dots come up in all 4 tosses}$
2. $B = \text{the set containing elements of } \Omega \text{ where 3 dots come up in at least one of the 4 tosses}$

VI. Prove the following statements. Provide a justification for each line of your proof.

1. Prove by mathematical induction: If \mathcal{C} is closed under the union of 2 sets then it is closed under finite union.
2. Prove using the definition of equality of two sets and without using the properties of set operations:

$$\text{If } A \text{ and } B \text{ are sets then } A \cup (A \cap B) = A$$

3. Prove de Morgan's Law: $\left(\bigcap_{\lambda \in \Delta} A_\lambda\right)^c = \bigcap_{\lambda \in \Delta} A_\lambda^c$

4. Prove using the properties of set operations:

$$\text{If } A \text{ and } B \text{ are sets then } (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

5. Prove using the definition of a sigma-field:

If \mathcal{F} is a sigma-field then it is closed under countable intersection.

VII. Disprove the following statements by providing a counterexample. Use $\Omega = \{a, b, c, d\}$. Let A , B and C be sets.

1. If $A \cup B = A \cup C$ then $B = C$.
2. If $B = A \cup C$ then $A = B - C$.